Modeling of Arterial Blood Flow

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ABSTRACT

Blood flow is a study of measuring the blood pressure and finding the flow through the blood vessel. Blood flow problem has been studied for centuries where one of the motivations was to understand the conditions that may contribute to high blood pressure. This occurs when the blood vessel became narrowed from its normal size. This paper presents a simple mathematical modeling of the arterial blood flow which was derived from the Navier-Stokes equations and based on some assumptions. A system of nonlinear partial differential equations for blood flow and the cross-sectional area of the artery were obtained. Finite difference method was adopted to discretized the equations. A linearized system of equations is obtained and solved numerically using two approaches which are the method of variation of parameters and a built-in function in Matlab called ODE45 which is based on Runge-Kutta. The results obtained showed that the latter method is a better method to apply for such system which is ODE45. Based on results that were obtained from this approach, we found that cross-sectional area does affect the blood flow in the arteries.

ABSTRAK

Pengaliran darah merupakan suatu kajian mengenai tekanan darah dan mengetahui aliran melalui saluran darah. Masalah pengaliran darah ini telah dikaji selama berkurun yang mana salah satu pendorong adalah untuk mengkaji faktor utama yang menyumbangkan kepada masalah penyakit tekanan darah tinggi. Keadaan ini terjadi disebabkan pengecilan saluran darah. Dokumen ini mengutarakan satu model matematik bagi pengaliran darah yang mudah yang diterbitkan daripada persamaan Navier-Stokes dan berdasarkan beberapa andaian. Satu sistem persamaan pembezaan separa tak linear untuk pengaliran darah dan luas bahagian rentas arteri dihasilkan. Kaedah beza terhad digunakan untuk meng'discretize'kan persamaan tersebut. Satu sistem yang linear akan terhasil dan diselesaikan mengunakan dua kaedah iaitu kaedah parameter bervariasi dan satu fungsi yang terdapat di dalam Matlab yang bernama ODE45. Hasil daripada dua kaedah tersebut dibanding dan kaedah terbaik telah dipilih iaitu ODE45. Daripada hasil model tersebut, kita dapati pengaliran darah dipengaruhi oleh saluran darah.

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ρ	-	Density
t	-	Time
L	-	Length
P	-	Pressure
v	-	Kinematic Viscosity
u	-	The components of velocity in axial (z) directions
w	-	The components of velocity in radial (r) directions
η	-	Radial Coordinate
R	-	Inner radius of the vessel
S	-	Cross-sectional Area
Q	-	Blood flow
$\frac{\partial P}{\partial z}$	-	Pressure Gradient

CHAPTER 1: INTRODUCTION

1.1 What is blood flow?

Blood flow research is very important for human health. According to Everett (2003), blood flow is defined as a study of measuring blood flow and pressure. The researches study the blood flow through arteries and veins. Wilkie (2003) in her study on Human blood flow measurement and modeling, states that the measure of oxygen and the nutrient concentration in blood is related with the blood flow.

1.2 Blood as fluid

Fluid is always mistaken as water or liquid. But actually, fluid can be anything that is not solid. Fluid is divided into gases and liquid. Brainydictionary.com define fluid as something that have freely move particles and change their relative position with no separation of the mass, and which simply yield to pressure. So, blood can be considered as fluid.

There are two categories of fluid which are Newtonian and non-Newtonian. From *Newtonian Fluid* (2006), Newtonian fluid is "fluid in which shear stress is linearly proportional to the velocity gradient in the direction perpendicular to the plane of shear".

On the other hand, non-Newtonian fluid is defined as "fluid in which the viscosity changes with the applied shear force". Most sources categorized blood as a non-Newtonian fluid. But according to Wilkie (2003), under some circumstances, for example those imposed on this problem, it is better to assume that blood is a Newtonian fluid.

1.3 Hypertension

Hypertension is also called high blood pressure where the blood pressure in the blood vessels is higher than normal pressure. Hyperdictionary.com defined hypertension as a situation when blood flow that getting through the vessels is greater than usual force. Simon (2002) states that there are two major factors that cause the hypertension that is:

i) When heart pumps blood with excessive force

ii) When blood vessels become narrow

There are two numbers to describe blood pressure that is the systolic pressure and the diastolic pressure. Systolic pressure is the pressure in arteries when the heart beats. Diastolic pressure is the pressure in arteries when the heart relaxes between beats. The blood pressure is high if the systolic pressure is higher than 140mmHg or the diastolic pressure is higher than 90mmHg. If the systolic pressure is less than 100mmHg or the diastolic pressure is less than 70mmHg, the blood pressure is low or hypotension.

1.4 Navier-Stokes equations

Navier (1785-1836) and Stokes (1819-1903) have established a system of equations, which is known as Navier-Stokes equations, to describe fluid flow motion. Since blood is a type of fluid, the flow is modeled using the same system of equations. Navier-Stokes equations consist of continuity equation and momentum equation (*Aeronautics - Fluid Dynamics - Level 3 Flow*, 2004).

1.4.1 Derivation of Continuity Equation

Consider the elemental volume's side of length is Δx, Δy and Δz as in Figure 1.1 below.

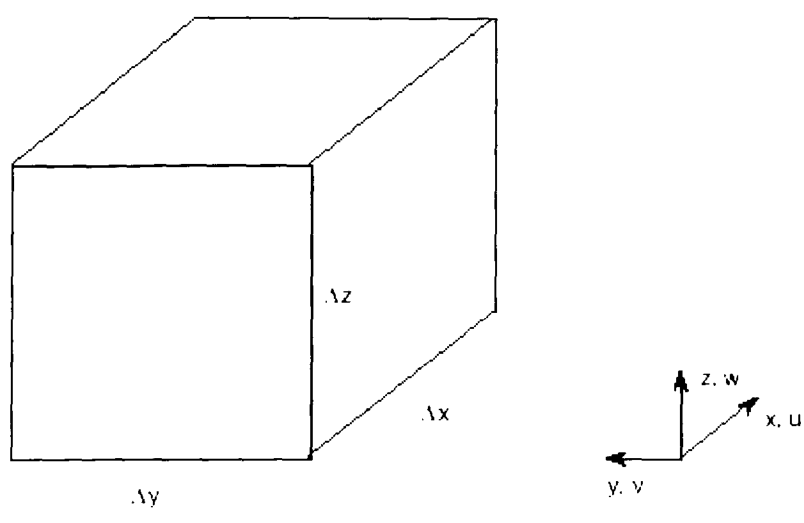

Figure 1.1: Elemental Volume

The rate of the mass for this elemental volume is depending on the rate mass of the fluid getting in and out through the faces. Mass flux in is product of density (ρ), fluid velocity (u) and the face area.

Mass flux in(x-direction) = $\rho u \Delta y \Delta z$,

Mass flux in(y-direction) = $\rho v \Delta x \Delta z$,

Mass flux in(z-direction) = $\rho w \Delta x \Delta y$.

And mass flux out is the same with mass flux in except that the density and the velocity may change as the fluid is getting through the volume. So, we say that the changes are in small quantities.

Mass flux out(x-direction) = $-(\rho + \Delta\rho)\ (u + \Delta u)\Delta y \Delta z$,

Mass flux out(y-direction) = $-(\rho + \Delta\rho)\ (v + \Delta v)\Delta x \Delta z$,

Mass flux out(z-direction) = $-(\rho + \Delta\rho)\ (w + \Delta w)\Delta x \Delta y$.

The negative sign in the mass is due to the fact that the mass in existing the elemental volume. The summation of the masses will result in the total mass of fluid accumulating in the volume.

$$\left(\frac{\Delta\rho}{\Delta t}\right)\Delta x \Delta y \Delta z .$$

So, the equation will be

$$z\text{-direction}: \quad -\left(\rho wu + \frac{\partial}{\partial z}(\rho wu)\Delta z\right)\Delta x\Delta y \quad .$$

Then we add all these terms together as expressed in the law of the conservation of momentum to obtain

$$\left(\frac{\partial}{\partial t}\right)(\rho u)(\Delta x\Delta y\Delta z) = \rho uu\Delta y\Delta z + \rho vu\Delta x\Delta z + \rho uu\Delta x\Delta y - \left(\rho uu + \frac{\partial}{\partial x}(\rho uu)\Delta x\right)\Delta y\Delta z$$
$$-\left(\rho vu + \frac{\partial}{\partial y}(\rho vu)\Delta y\right)\Delta y\Delta z - \left(\rho wu + \frac{\partial}{\partial z}(\rho wu)\Delta z\right)\Delta x\Delta y + \Sigma F_x$$

where ΣF_x is summation of all external forces in the control volume. Then, we do some calculations and moving some terms. Then we get

$$\left(\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) + \frac{\partial}{\partial z}(\rho wu)\right)\Delta x\Delta y\Delta z = \Sigma F_x \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

From the Product Rule, we get

$$\left(\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} + u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z}\right)\Delta x\Delta y\Delta z = \Sigma F_x .$$

Since the last four terms in parentheses are the continuity equation times u that is equal to 0 and that leaves us with

$$\left(\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z}\right)\Delta x\Delta y\Delta z = \Sigma F_x .$$

So, using the same way, we can get the force for y and z direction

$$\left(\rho\frac{\partial v}{\partial t}+\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}\right)\Delta x\Delta y\Delta z=\Sigma F_y,$$

$$\left(\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z}\right)\Delta x\Delta y\Delta z=\Sigma F_z.$$

After we get the equations of momentum change and flux, now we can find the equation of the forces on the control volume that is body forces and surface forces. Body force is due to gravity

$$g_x\rho\Delta x\Delta y\Delta z$$

and the force that acting in x-direction due to the direct stresses (σ_{ij}) is

$$-\sigma_{xx}\Delta y\Delta z \text{ and } \left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}\Delta x\right)\Delta y\Delta z.$$

Add these two forces

x-direction : $\frac{\partial\sigma_{xx}}{\partial x}\Delta x\Delta y\Delta z$,

y-direction : $\frac{\partial\sigma_{yx}}{\partial y}\Delta x\Delta y\Delta z$,

z-direction : $\frac{\partial\sigma_{zx}}{\partial z}\Delta x\Delta y\Delta z$.

The stress σ_{xx} includes the pressure and the normal viscous stress τ_{xx}. The force in the x-direction as

$$\left(-\frac{\partial \rho}{\partial t}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)\Delta x \Delta y \Delta z .$$

The viscous stresses are

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x},$$

$$\tau_{yx} = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),$$

$$\tau_{zx} = \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right).$$

Then, we combine all terms and the body forces and arrange the equation, we will get

$$\left(\rho g_x - \frac{\partial \rho}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right)\Delta x \Delta y \Delta z .$$

With all the equation, put together; we will get the final equation for x-direction

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial x}+\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x \quad . \tag{2}$$

For y-direction, the equation is

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial y}+\eta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y \quad , \tag{3}$$

and for z-direction

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\eta\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z \quad . \tag{4}$$

The system of equations (1) to (4) is the unsteady, incompressible 3D Navier-Stokes equations.

1.5 Problem Statement

There is several researches that had already been done on blood flow problem such as Wilkie (2003) on human blood flow measurement and modeling, Salmon, Thiriet and Gerbeau (2002) on his study on Numerical simulations of blood flows in arteries for interventional medicines, Bitsch (2002) on blood flow in microchannels and many more. Most of the research studies are about the flow in arteries and veins. In some conditions, arteries can become smaller and it may have an effect on the blood flow and the pressure. This study will focus on blood flow in arteries during diastole which is the heart is at rest and how does the blood vessel may affect the blood flow.

1.6 Project's Objectives

The objectives of this project are to:

i) model the blood flow problem mathematically using the Navier-Stokes equations,

ii) compare and contrast on numerical methods that best to solve the obtained governing equations,

iii) decide on which factor that contributes to hypertension based from the numerical result obtained,

v) analyze the result and decide what condition may affect the blood flow and blood pressure

1.7 Methodology

The project is a mathematical modeling nature. Therefore the methodology will be based on the modeling process which has 7 phases. The phases of the modeling process are shown in Figure 1.2.

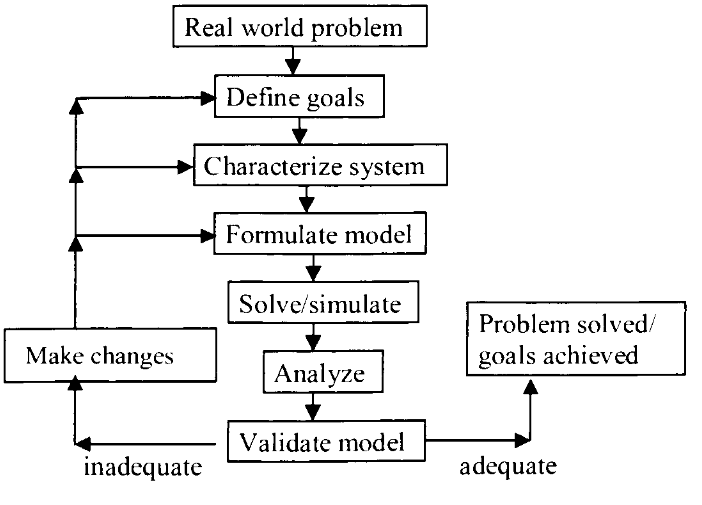

Figure 1.2: Mathematical modeling process

In real world phase, the problem of the real world will be studied and identified. Next, goal of the project will be defined as the objective the project will be listed out. Then, in the characterize system phase, the system of the problem will be studied and all the assumption that required to solve the problem will be determined. Next, after the problem has been studied and the assumptions required have been made, the model that represents the problem will be formulated based on all the assumptions were. The model will be solved and simulated in order to get the result so that the result can be analyze. After analyzing the result, we can determine weather the model that has been formulated is valid or not. If the model is not valid with the problem, changes will be made to the model. Problem is solved if the model is valid to the real problem.

1.8 Scope

The scopes of this project are:

i) Modeling the blood flow in arteries

ii) Simulate the blood flow in arteries

1.9 Contribution of the project

This project will make a contribution in providing a simple blood flow mathematical model which will be simulated to analyze and determine what the conditions are and factors that affect the blood flow of in human arteries. From the algorithm, we can see the conditions of blood vessel that may cause an abnormal blood flow.

1.10 Expected Outcome

The expected outcome from this project is a system or a program that simulate the blood flow at normal condition and abnormal condition. From the results, we can conclude if the arteries size affects the blood flow and the pressure.

1.11 The Structure of the Report

In the first chapter, general information regarding the project is described. This chapter introduces and describes blood flow problem, fluid in general, hypertension and the derivation of the Navier-Stokes equations. Other than that, the problem that is focused in the project, its objective, methodology used, the scope of the project and the expected outcome will be stated in this chapter.

In the following chapter, the background and the literature review of blood flow studies are discussed. We will look at the previous researches and study on the blood flow that has been conducted by other researchers.

In chapter 3, the mathematical model for the blood flow problem will be developed based on assumptions and boundary conditions.

Next, chapter 4, we will discuss about numerical methods that are chosen to solve the problem. Based on those numerical methods, algorithm for the system will be constructed.

In the fifth chapter, the result that obtained from the system will be analyzed. By that, we can see what are the conditions that affect the blood flow. Besides that, we will compare result that obtained from all numerical methods that we have chosen previously.

In the final chapter, we will conclude the project and state the future work that can be done to improve this report.

CHAPTER 2: BACKGROUND

2.1 Introduction

In solving blood flow problem, there are some steps that need to be followed. Firstly, the governing equations and the boundary conditions will be determined. The governing equations that will be used are the output from the mathematical modeling which is derived from the Navier-Stokes equations based on some assumptions. Secondly, the numerical solutions that will be used for solving the problem are determined. There are several numerical solutions that can be used. If the equations that we obtained are partial differential equation (PDE) form, we can use CFD solver or Numerical methods to solve it. The examples for CFD solvers are CFD Studio, FLUENT and CFD ACE. The examples of numerical methods that can be used to solve partial differential equation are Finite Element Method, Finite Volume Method, and Finite Difference Method. Meanwhile, if the equations are in ordinary differential equation (ODE) form, we can solve the equations using any mathematical package like MatLab, Maple or theoretically. In this chapter, we will review some of the research papers that have been published on blood flow problem.

2.2 Selected Work on the Blood Flow Problem

There are many researches done on blood flow. Most of the researches focused on the time averaged rates of flow and pressure and the properties of the arterial system. In the 17^{th} century, Hales, Poiseuille and Young are possibly the first few persons to study the properties of blood flow (Lin, 2004). From Reverend Hales study, he stated two fundamental qualitative observations. His first observation is the elasticity of the large arteries was responsible for the continued way out of blood from the tangential arteries during ventricular diastole. His second observation is the main resistance to flow out of the arteries resided in the capillaries. In the middle of 18^{th} century, Harvey and Hales believed that the pressure arose at the same time in all the arteries. Meanwhile, in early 1775, Euler believed that ventricular contraction set up a wave which was propagated through the arteries with a limited velocity. Until today, researches and studies on blood flow are still on going.

Wilkie (2003) focused on the discussion of blood flow, blood pressure and cardiovascular system and the regulation of the cardiac output. She also talked about the existing method for measuring blood flow and blood pressure using some devices. These physical methods discussed in her paper are Indicator-dilution Methods, Electromagnetic flowmeters, Ultrasonic flowmeters, Plethysmography. According to Wilkie(2003) there are two major components in the circulatory system that are the cardiovascular system and the lymphatic system. The cardiovascular system consists of blood, blood vessels and the heart.

In the modeling part, Wilkie (2003) stated a few assumptions. She assumed that flow in arteries is a continuum and blood is a Newtonian fluid. To solve the problem, she used Navier-Stokes equation because the equations explain the axial symmetric motion.

From the equations, she then simplifies it into an ordinary differential equation and then solved the ordinary differential equation by using Bessel functions.

Heys et al. (2004) studied the blood flow in large vessels by using Navier-Stokes equations for the fluid domain and elasticity equations for the vessel wall. The governing equation obtained is a partial differential equation which can be solved using an iterative method. On that paper, they presented another approach that based on multilevel minimization of the finite element approximation error using a Least Squares (LS) norm. Besides that, they also make a comparison between the LS finite element approach and other numerical methods especially the commercial package CFD-ACE.

In the same paper, the authors address three mathematical issues related with coupled problems. First, the equations for fluid or solid are often nonlinear. Second, "standard discretizations of the linearized Navier-Stokes and the elasticity equations yields a matrix that is not positive definite, resulting in a linear system that is difficult to solve iteratively". Finally, the stresses matching between the solid and fluid are critical for a precise solution to the coupled problem.

The authors stated that First-Order System Least Square (FOSLS) finite element formulation is a good way to model blood flow through compliant vessels and it has three advantages compared to other approaches: