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Modeling of Axial Flow between Eccentric Cylinders

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Abstract. The study of thread annular flow is motivated due to the fact that thread injection is a promising method for placing medical implants within the human body with minimum surgical trauma. The porous thread is stored on a spool and injected within a fluid by applying a pressure gradient. The injection process must be smooth, and lateral deviations of the thread are undesirable. This paper presents a mathematical modeling of the problem and discusses preliminary findings concerning the nature of the basic flow.

Keywords: mathematical modeling, axial flow, eccentric cylinders, thread-annular flow

1. Introduction

The injection of fluid into the body using a needle or syringe is an important application of fluid dynamics. Through modeling the problem mathematically, a better understanding of the way in which key parameters, such as the speed of injection, affect the fluid flow characteristics within the syringe can be obtained. It is hoped that with this investigations, the injection can be carried out more proficiently and the pain of the patient can be reduced. In the surgical process known as thread injection a surgeon not only injects fluid into the body, but also various medical implants. A particular application of chemical thread injection is in lip augmentation found in most plastic surgery industries.

A simplified version of the injection process can be modeled mathematically by considering the axial flow between concentric cylinders with the inner cylinder (representing the thread) moving at a constant velocity. The first study was that of Frei et. al. [1] who examined the problem experimentally and theoretically. Walton [2] considered the stability of the basic thread-annular flow and reported that there are discrepancies between the theoretical and experimental results. All previous theoretical studies assume the thread to remain in a concentric position, although Frei et al's [1] experiments indicated that the effects of eccentricity could be significant. In this study, the thread is allowed to occupy a fixed eccentric position within the pipe with the aim of studying mathematically the effect of this eccentricity on the basic flow.

2. Mathematical Formulation

The porous thread is stored on a spool and injected within a fluid by applying an axial pressure gradient to the cylindrical container holding the liquid and the thread. Since it is desirable to have smooth injection, the flow in the region between the cylinders needs to be kept laminar. The flow in this region is assumed to be fully-developed, i.e. it is independent of both time and axial distance. The same cylindrical polar coordinate system (x^*, r^*,θ) = (a^*x, a^*r,θ), with r = 0 the axis of the outer tube, used in [2] will be adopted. Here, x, r, and θ represent the non-dimensional axial, radial and azimuthal coordinates and the tube is of dimensional radius a^*. The velocity components are written as (u^*, v^*, w^*) = ($g^*a^{*2}/\rho^* \nu^*$)(u, v, w), where $-4g^*$ is the constant axial pressure gradient to be applied to the pipe. ρ^* and ν^* are the density and kinematic viscosity of the incompressible fluid. The pressure p^* and time t^* are expressed in the form ($g^{*2}a^{*4}/\rho^* \nu^{*2}$)$p$ and ($g^*a^*/\rho^* \nu^*$)t respectively. Using these scalings in the unsteady Navier-Stokes equations we obtained the system of equations in a non-dimensional form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{1}{r}\frac{\partial w}{\partial \theta}=0, \tag{1.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial P}{\partial x}+\frac{1}{R}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right], \tag{1.2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \theta}-\frac{w^2}{r}=-\frac{\partial P}{\partial r}+\frac{1}{R}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}-\frac{v}{r^2}-\frac{2}{r^2}\frac{\partial w}{\partial \theta}\right], \tag{1.3}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \theta}+\frac{vw}{r}=-\frac{\partial P}{\partial \theta}+\frac{1}{R}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{w}{r^2}+\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right]. \tag{1.4}$$

where the Reynolds number R is defined by $R=\dfrac{g^*a^{*3}}{\rho^*\nu^{*2}}$.

The thread injection process is modeled by considering the axial flow between cylinder with r = 1 representing the syringe and $r = \delta$ representing a thread of dimensional radius $\delta = a^*\delta$ placed in an eccentric position within the syringe. The center of the thread is now at position $z = \varepsilon$, y = 0 where y and z are coordinates in the cross-section of the pipe as shown in Figure 1. Then the boundary of the thread is described to order ε by $r = \delta + \varepsilon \cos\theta$ (obtained by using the cosine rule) where $\varepsilon << \delta$ is the parameter that controls the eccentricity of the thread position. This is clearly shown in Figure 2 which is an enlargement of part of Figure 1. If ε = 0 then the thread is in a concentric position [2]. The thread is moving in the axial direction with non-dimensional velocity V (corresponding to the dimensional velocity V^* = ($g^*a^{*2}/\rho^* \nu^*$)V = (ν^*/a^*)RV. The basic flow is assumed to be steady and unidirectional so a solution of the Navier-Stokes equations (1.1)-(1.4) is written in the form $u=u_0(r)+\varepsilon\cos\theta u_1(r)+...$ and $v = w = 0$, subject to the fixed axial pressure gradient in the non-dimensional form $\frac{\partial p}{\partial x}=-\frac{4}{R}$.