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On Ordered Weighted Averaging Operator and Monotone Takagi-Sugeno-Kang Fuzzy Inference Systems

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***Abstract*—The necessary and/or sufficient conditions for a Takagi-Sugeno-Kang Fuzzy Inference System (TSK-FIS) to be monotone has been a key research direction in the last two decades. In this paper, we firstly define fuzzy membership functions (FMFs) with single and continuous *support;* and consider TSK-FIS with a "*grid partition*" strategy for computing its firing strengths with product T-norm (here after denoted as TSK-FIS-product). We also define a more general *joint* necessary condition, whereby each constituent itself is a necessary condition for the TSK-FIS-product model. The first necessary condition indicates that the normalized *firing strength* must not be indeterminate (i.e., 0/0), i.e. susceptible to the "tomato classification problem". The second necessary condition indicates that all *restricted* consequents of fuzzy if-then rules must be defined. Based on the principle of the *ordered weighted averaging (OWA) operator* as well as the concept of increasing *orness* in OWA and *hyperboxes*, a general joint *sufficient condition* for a TSK-FIS-product model to be monotone is derived. Three case studies of the developed methods for undertaking Failure Mode and Effect Analysis (FMEA) and image processing tasks are presented. The results are compared, analyzed, and discussed, demonstrating the usefulness of our developed methods.**

***Index Terms*—Takagi-Sugeno-Kang Fuzzy Inference System, monotone, necessary condition, sufficient condition, ordered weighted averaging operator, increasing *orness*, *hyperboxes,* FMEA, edge detection.**

I. INTRODUCTION

A. Background

A Fuzzy Inference System (FIS), denoted as f, is known as a monotone non-decreasing FIS, if it is a mapping $f: X \rightarrow Y$, that satisfies $f\left(\boldsymbol{x}_{(1)}=\left(x_{1,(1)},\dots,x_{i,(1)},\dots,x_{n,(1)}\right)\right) \leq f\left(\boldsymbol{x}_{(2)}=\left(x_{1,(2)},\dots,x_{i,(2)},\dots,x_{n,(2)}\right)\right)$ for all $x_{i,(1)} \leq x_{i,(2)} \in X_i$, $i \in \{1,\dots,n\}$, in an n-dimensional input space $X \in \mathbb{R}^n$ and an output space $Y \in \mathbb{R}$. The importance of the monotone property as a *prior* condition for FIS modelling has been highlighted [1]-[16]. In general, research on monotone-preserving FIS (hereafter denoted as monotone FIS) models encompasses three key aspects: (i) mathematical conditions for various FIS variants to satisfy the monotone property [1]-[16], e.g. in Takagi-Sugeno-Kang FIS (TSK-FIS) [4] and interval-type-2 FIS [11] models; (ii) methods to construct monotone FIS models, either via expert knowledge (knowledge-driven) or data samples (data-driven) [8][14]-[16]; and (iii) application of monotone FIS models to different domains [5][7][17]-[21], including the use of TSK-FIS as *n-ary aggregation functions* [5]. The popularity of monotone FIS models has also been inspired by monotone radial basis function networks [22].

In the current literature, however, most studies on the mathematic conditions of TSK-FIS are focused on the *sufficient condition*. In addition, studies on the *sufficient condition* are generally confined to classical fuzzy membership functions (FMFs) such as triangular, trapezoid, and Gaussian. The *sufficient condition* usually considers consequents with real numbers or monotone functions. As indicated in [2][3], the relationship between the sufficient condition and the necessary condition (if it exists) is seldom discussed, which remains as a research gap [4].

On the other hand, the Ordered Weighted Averaging (OWA) operator [23][24] offers a powerful and flexible information fusion technique. While a monotone FIS model is related to the OWA principle, their relationship is rarely studied from the FIS perspective [25]. An m-ary OWA is a mapping of an m-dimensional weight vector $\boldsymbol{w}=(w_1,\dots,w_m)^T$ with the properties of $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j = 1$, such that $OWA(a_1,\dots,a_m)=\sum_{j=1}^{m} w_j b_j$, where b_j is the jth largest a_j. Both $(a_1,\dots,a_m)$ and $(b_1,\dots,b_m)$ are vectors with m real numbers, which can be either positive, zero, or negative real values. Vector $(b_1,\dots,b_m)$ is also an ordered vector. OWA operators have wide applicability owing to their flexibility in modelling a family of parameterized averaging aggregation functions, in conjunction with the *orness* concept [26][27] and orness measure of OWA [24]. The orness measure of $\boldsymbol{w}=(w_1,\dots,w_m)^T$ increases, i.e., $Or(\boldsymbol{w}_{(1)}) > Or(\boldsymbol{w}_{(2)})$, if two m-dimensional weight vectors, i.e., $\boldsymbol{w}_{(1)}$ and $\boldsymbol{w}_{(2)}$, satisfy $\boldsymbol{w}_{(1)}=(w_{1,(1)},\dots,w_{m,(1)})^T$ and $\boldsymbol{w}_{(2)}=(w_{1,(1)},\dots,w_{\xi}-\epsilon,\dots,w_{\rho}+\epsilon,\dots,w_{m,(1)})^T$, where $\epsilon > 0$ and $\xi < \rho$ [26]. The use of orness for partition design in data pre-processing tasks has been highlighted in [28]. In addition, many research studies on *hyperboxes* in fuzzy systems and neural networks are available in the literature [29]. Both concepts of orness and hyperboxes are exploited in this study.

B. Research Aim

The aim of this research is three-fold. Firstly, we embed a "*grid partition*" strategy [30] into TSK-FIS to compute the firing strengths with product T-norm (here after denoted as TSK-FIS-product). We show that a normalized firing strength for a TSK-FIS-product model can be *indeterminate* (i.e., $0/0$), if the model is *sparse* [31][32] i.e., the so-called *tomato classification problem* [33]. This issue forms the fundamental challenge in fuzzy rule interpolation [16][34][35][36].

To facilitate our analysis, a number of settings are clarified: (i) FMFs with *single* and *continuous support* are defined and

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used; (ii) a hyperbox is formed by the antecedent of a fuzzy if-then rule, which adopts FMFs with single and continuous support; (iii) a new *restriction* for the functional consequents of the TSK-FIS-product model is defined, whereby the restriction of each functional consequent is a subspace of $\mathbb{R}^n$. In addition, we expand a multi-input TSK-FIS-product model as a series of single-input TSK-FIS-product-*like* models to facilitate our analysis.

The second research aim is to establish several new theorems associated with the design and construction of a monotone TSK-FIS-product model. Specifically, we define a new and more general *joint* necessary condition, whereby each constituent itself is a necessary condition, to ensure a TSK-FIS-product model always produces valid outputs for all $\boldsymbol{x}$. We provide counter examples to illustrate that some conditions, which constitute part of a joint sufficient condition, are not the necessary condition, although they have been practically used to design monotone TSK-FIS-product models.

We further consider an additional condition whereby the maximal of two FMFs overlap each other. In short, instead of focusing on FMF designs, we scrutinize the normalized firing strength of an FMF (see Property 3) in our analysis. Inspired by the OWA principle, i.e., the concepts of increasing orness [26], as well as the hyperbox structure [29], a more general *joint* sufficient condition, which comprise of two joint necessary conditions, for establishing monotone TSK-FIS-product models are formulated. It allows monotone TSK-FIS-product models to be designed with varying FMFs comprising single and continuous support, either convex or non-convex, normal or sub-normal, or with jump discontinuity. Our developed method also considers fuzzy rule consequents with non-monotone functions.

Three case studies pertaining to Failure Mode and Effect Analysis (FMEA) [37] and image processing are conducted. The results illustrate the usefulness of our sufficient condition for formulating a set of feasible solutions. From the image processing case study, our results are useful for developing monotone TSK-FIS-product models to serve as a *blending function* in image edge detection [38].

C. Contributions

This research makes the following major contributions:

1. A joint necessary condition for establishing monotone TSK-FIS-product models is formulated. The first condition indicates that the normalized firing strength of a TSK-FIS-product model must not be indeterminate (i.e., 0/0). The second condition imposes that all restricted functional consequents of fuzzy if-then rules must be defined. Note that each constituent of the joint necessary condition is itself a necessary condition.
2. Inspired by increasing orness in the OWA principle and the hyperbox structure, a new proposition, along with its proof, to serve as a new joint sufficient condition for constructing monotone TSK-FIS-product models is formulated. It covers the following two overlapped FMF cases (i.e., FMFs with single and continuous support), which are yet to be analyzed in the existing sufficient conditions pertaining to monotone TSK-FIS-product models. Specifically, by imposing a restriction that the maximal of two FMFs overlap each other, we allow
   a. convex, non-convex, discontinuous and/or sub-normal fuzzy sets, and their mixture, as options to construct a monotone TSK-FIS-product model.
   b. non-monotone functional consequents, which are monotone within the associated restriction, as options to construct a monotone TSK-FIS-product model.
3. To better explain our analysis, we define a number of new and useful notions for analyzing TSK-FIS models in general, as listed in the second paragraph of Section I (B).

Our derived joint sufficient condition provides a method to systematically design monotone TSK-FIS-product models, either manually or automatically based on data-driven techniques).

The rest of this paper is organized as follows. In Section II, a TSK-FIS-product model is defined. We analyze the normalized firing strength of the TSK-FIS-product model to be indeterminate. In Section III, FMFs with single and continuous support are defined and analyzed. This leads to the definition of hyperboxes formed by the antecedents of fuzzy if-then rules. In Section IV, an analysis of a TSK-FIS-product model adopting FMFs with single and continuous support is presented. In Section V, a new expansion pertaining to the TSK-FIS-product model is described. The major results of this study are explained in Section VI. Three case studies of our developed methods are presented in Section VII. Concluding remarks and suggestions for further research are given in Section VIII.

II. TAKAGE-SUGENO-KANG FUZZY INFERENCE SYSTEM

Consider a TSK-FIS-product model with an input domain $X_i \in X \in \mathbb{R}^n$, $i \in \{1, \dots, n\}$, the input variable x_i is partitioned into $p_i \geq 1$ FMFs. Each partition is tagged with a linguistic term, $A_i^{r_i}$, with its corresponding FMF $\mu_i^{r_i}(x_i)$, where $r_i \in \{1, \dots, p_i\}$, and $r_1, \dots, r_n$ is an integer. To simplify the explanation, a numeral v is used to label each fuzzy if-then rule, such that $v(r_1, r_2, \dots, r_n) = [\sum_{i=2}^{n}((r_i - 1) \times (\prod_{g=1}^{i-1} p_g))] + r_1$ and $v \in \{1, \dots, P\}$, where $P = \prod_{i=1}^{n} p_i$. If $n = 1$, a single-input TSK-FIS-product model is expected, and $v = r_1$. If $n > 1$, a multi-input TSK-FIS-product model is expected. The v^{th} fuzzy if-then rule of a TSK-FIS-product model is as follows:

R^v: IF x_1 is A_1^v AND ... AND x_n is A_n^v THEN y is $y^v(\boldsymbol{x})$.

The firing strength of a fuzzy rule, i.e., $R^v: A^v \rightarrow y^v(\boldsymbol{x})$, is defined as follows.

Definition 1. Given a real-valued input vector $\boldsymbol{x}$ and the fuzzy membership value of $A_i^{r_i}$ is represented by $\mu_i^{r_i}(x_i)$, the *firing strength* and *normalized firing strength* of the v^{th} fuzzy rule can be obtained using (1) and (2), respectively.

$$A^v(\boldsymbol{x}) = \prod_{i=1}^{n} A_i^{r_i}(x_i) \quad (1)$$

$$\overline{A^v(\boldsymbol{x})} = \frac{A^v(x)}{\sum_{v=1}^{P} A^v(x)} \quad (2)$$

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A TSK-FIS-product model is defined as follows.

Definition 2. A TSK-FIS-product model is a mapping of $f: X \rightarrow Y$, with P fuzzy if-then rules, i.e., $R^v: A^v \rightarrow y^v(\boldsymbol{x})$, which can be obtained using Eq. (3).

$$f(\boldsymbol{x}) = \frac{\sum_{v=1}^{P} A^v(x) \times y^v(x)}{\sum_{v=1}^{P} A^v(x)} = \sum_{v=1}^{P} \overline{A^v(\boldsymbol{x})}\, y^v(\boldsymbol{x}) \quad (3)$$

The characteristics of $A^v(\boldsymbol{x})$ and $\overline{A^v(\boldsymbol{x})}$ from Eqs. (1) and (2), respectively, are explained, as follows.

Property 1. Characteristics of $A^v(\boldsymbol{x})$ and $\overline{A^v(\boldsymbol{x})}$.

P1.1. $A^v(\boldsymbol{x}) = 0$ if there exists an i-th element such that $A_i^{r_i}(x_i) = 0$.

P1.2. If $\sum_{v=1}^{P} A^v(\boldsymbol{x})$=0, then $\overline{A^v(\boldsymbol{x})}$ is indeterminate as 0/0 is undefined.

P1.3. If $\sum_{v=1}^{P} A^v(\boldsymbol{x})$>0 is true for all $\boldsymbol{x}$, then $\sum_{v=1}^{P} \overline{A^v(\boldsymbol{x})}$ =1, and $0 \leq \overline{A^v(\boldsymbol{x})} \leq 1$ are always true.

III. Fuzzy Membership Functions with Single and Continuous Support

A. Support of Fuzzy Membership Functions

A *strong α-cut* [30] of $\mu_i^{r_i}(x_i)$ is defined as $\mu_{i,\alpha}^{r_i\prime}(x_i)$ =$\{x_i | \mu_i^{r_i}(x_i) > \alpha\}$, where $\alpha \in [0,1]$. With a strong α-cut, the *support* of $\mu_i^{r_i}(x_i)$ and an interval representing the support are presented in Property 2.

Property 2. The *support* of convex and non-convex $\mu_i^{r_i}(x_i)$

P2.1. The support of $\mu_i^{r_i}(x_i)$ is defined as $\mu_{i,0}^{r_i\prime}(x_i)$ =$\{x_i | \mu_i^{r_i}(x_i) > 0\}$. (see pp. 18 in [30])

P2.2. The support of convex $\mu_i^{r_i}(x_i)$ is denoted as a *single* and *continuous* interval, i.e., $[\underline{x_i^{r_i}}, \overline{x_i^{r_i}}]$, such that $[\underline{x_i^{r_i}}, \overline{x_i^{r_i}}] \in X_i$ (see Fig. 1(a)).

P2.3. An *uncommon* $\mu_i^{r_i}(x_i)$ exists such that $\mu_i^{r_i}(x_i) = 0$ for all x_i (which still satisfies Property A.1 in Appendix). In this *uncommon* situation, the support does not exist.

P2.4. The support of non-convex $\mu_i^{r_i}(x_i)$ is usually a single and continuous interval too, i.e., $[\underline{x_i^{r_i}}, \overline{x_i^{r_i}}]$, such that $[\underline{x_i^{r_i}}, \overline{x_i^{r_i}}]$ $\in X_i$ (see Fig. 1(b)).

P2.5. The support of *uncommon* non-convex $\mu_i^{r_i}(x_i)$ could be discontinuous (see Fig. 1(c)).

Note that Property A in Appendix clarifies the possible FMFs in this study, i.e., convex/non-convex, normal/sub-normal, jump discontinuous, or complete FMFs.

B. A class of FMFs with Single and Continuous Support

In this study, we focus on a class of FMFs with single and continuous support, which is defined as follows. An example is given in Fig. 1.

Definition 3. An FMF $\mu_i^{r_i}(x_i)$ has *single* and *continuous* support, iff for any $d_1, d_2 \in X_i$, such that $\mu_i^{r_i}(d_1) > 0$, $\mu_i^{r_i}(d_2)$ >0, and any $\lambda \in [0,1]$, $\mu_i^{r_i}(\lambda d_1 + (1-\lambda) d_2) > 0$, is always true.

For any two points, $d_1, d_2 \in X_i$, such that $\mu_i^{r_i}(d_1) > 0$ and $\mu_i^{r_i}(d_2) > 0$, all their convex combinations can be obtained as $\lambda d_1 + (1-\lambda) d_2$, where $\lambda \in [0,1]$. In addition, an FMF with more than single support can practically be considered as comprising a few FMFs. Referring to the example in Fig. 1(c), the FMF can be treated as two FMFs, each with a single and continuous support. Remark 1 indicates the importance in defining the class of FMFs as in Definition 3.

Remark 1.

R1.1. All convex FMFs have either single and continuous support or no support.

Proof. Given an FMF with Property A.1, Property P2.2, or P2.3 exists.

R1.2. Not all non-convex FMFs have single and continuous *support*.

Proof. See Properties P2.4, P2.5, and Figs. 1(b) and 1(c).

R1.3. Triangular and trapezoidal FMFs have single and continuous support.

Proof. Since triangular and trapezoidal FMFs are convex, and $\mu_i^{r_i}(x_i)$ ≠0 for all x_i, following Remark R1.1, these FMFs have single and continuous support.

R1.4. All non-zero FMFs, either convex FMFs (e.g., Gaussian FMFs) or non-convex FMFs, have single and continuous support.

Proof. See Properties P2.2, P2.4, and P2.5

R1.5. FMFs with single and continuous support may not always be normal; they can be sub-normal.

Proof. Straightforward by following Property A.2.

R1.6. FMFs with single and continuous support may have *jump discontinuous* (in the sense of Property A.3).

Proof. Straightforward by following Property A.3.

Remark 1 indicates that many FMFs fall under the umbrella of Definition 3. As such, the analysis of our study has a wide coverage of FMFs options.

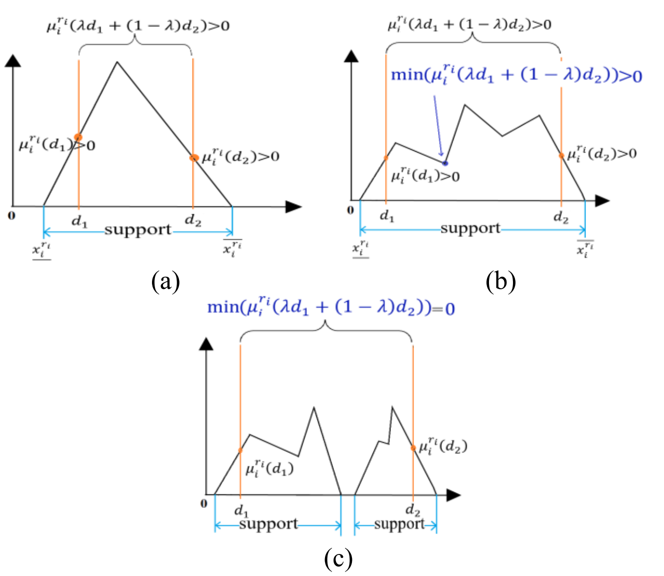

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Fig. 1. Examples of (a) a convex FMF with single and continuous support; (b) a non-convex FMF with single and continuous support; (c) an *uncommon* FMF with discontinuous supports (i.e., two supports), for any $d_1, d_2 \in X_i$.

C. An n-dimensional hyperbox formed by FMFs with Single and Continuous Support

In Definition 4, the support of A^v is defined as an n-dimensional hyperbox [29][39], denoted as HA^v. This definition is important for establishing Theorem 1 in Section IV-C. An example of a 2-dimensional hyperbox formed by two FMFs from Figs. 1(a) and 1(b) is shown in Fig. 2.

Definition 4. The support of A^v (denoted as $A_0^{v\prime}$) is defined as an n-dimensional hyperbox that is confined by its vertices, i.e., the minimum point $\underline{HA^v}$ and maximum point $\overline{HA^v}$. Both $\underline{HA^v}$ and $\overline{HA^v}$ are represented as two n-dimensional vectors, as follows.

$\underline{HA^v} = (\underline{x_1^{r_1}}, \underline{x_2^{r_2}}, \dots, \underline{x_n^{r_n}})$

$\overline{HA^v} = (\overline{x_1^{r_1}}, \overline{x_2^{r_2}}, \dots, \overline{x_n^{r_n}})$

such that $\overline{HA^v} \geq \underline{HA^v}$ and $A_0^{v\prime} \in X$.

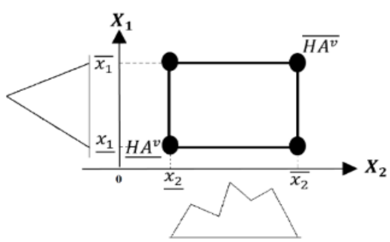

Fig. 2. An example of a 2-dimensional hyperbox $A_0^{v\prime} = (\underline{HA^v}, \overline{HA^v})$

IV. Analysis on a TSK-FIS-Product Model with FMFs of Single and Continuous Support

A. Analysis of the Antecedents of Fuzzy If-Then rules

The normalized firing strength, i.e., $\overline{A^v(\boldsymbol{x})}$ obtained from the antecedents of a set of fuzzy if-then rules, is analyzed. Property 1 leads to a specific *mapping* as in Definition 5 and Property 3, as follows.

Definition 5. A mapping, $T: X \to \mathcal{P}(\boldsymbol{x})$, where $\boldsymbol{x} \in X \in \mathbb{R}^n$ is considered, whereby $\mathcal{P}(\boldsymbol{x}) = (\overline{A^1(\boldsymbol{x})}, \overline{A^2(\boldsymbol{x})}, \dots, \overline{A^P(\boldsymbol{x})})$ is a vector with P values, $\overline{A^v(\boldsymbol{x})} \in \{\frac{0}{0}, [0,1]\}$ and $v \in \{1,2,\dots,P\}$.

Property 3. Mapping T has the following characteristics.

P3.1. Given a mapping T, $\mathcal{P}(\boldsymbol{x}_{(1)}) = \mathcal{P}(\boldsymbol{x}_{(2)})$, where $\boldsymbol{x}_{(1)} \neq \boldsymbol{x}_{(2)}$ can occur.

P3.2. Given a mapping T (denoted as $T_{(1)}$), it always exists another T (denoted as $T_{(2)}$), such that identical vectors of $\mathcal{P}(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$, can be obtained. This indicates $\mathcal{P}_{(1)}(\boldsymbol{x}) = \mathcal{P}_{(2)}(\boldsymbol{x})$, for all $\boldsymbol{x} \in X$, pertaining to $T_{(1)}$ and $T_{(2)}$ always occurs, respectively.

Property P3.2 is the reason for us to focus on the normalized firing strength, instead of FMFs design, in deriving the new joint sufficient condition. The importance of Property P3.2 can be observed in the example in Section VII-C. In addition, subject to the restriction that all FMFs must have single and continuous support (see Definition 3), Definition 6 is presented, as follows.

Definition 6. Given two antecedents, $A^{v,(1)}$ and $A^{v,(2)}$, the intersection of supports of $A^{v,(1)}$ and $A^{v,(2)}$ with an overlap is denoted as $A_0^{(v,(1)\cap v,(2))\prime} = A_0^{v\prime,(1)} \cap A_0^{v\prime,(2)}$, where $(r_{1,(1)}, r_{2,(1)}, \dots, r_{n,(1)}) \leq (r_{1,(2)}, r_{2,(2)}, \dots, r_{n,(2)})$. In other words, $A_0^{(v,(1)\cap v,(2))\prime}$ is defined as an n-dimensional hyperbox with its minimum point $\underline{HA^{v,(1)\cap v,(2)}}$ and maximum point $\overline{HA^{v,(1)\cap v,(2)}}$. Both $\underline{HA^{v,(1)\cap v,(2)}}$ and $\overline{HA^{v,(1)\cap v,(2)}}$ are represented by two n-dimensional vectors, as follows.

$\underline{HA^{v,(1)\cap v,(2)}} = (\underline{x_1^{r_{1,(2)}}}, \underline{x_2^{r_{2,(2)}}}, \dots, \underline{x_n^{r_{n,(2)}}})$

$\overline{HA^{v,(1)\cap v,(2)}} = (\overline{x_1^{r_{1,(1)}}}, \overline{x_2^{r_{2,(1)}}}, \dots, \overline{x_n^{r_{n,(1)}}})$

such that $\overline{HA^{v,(1)\cap v,(2)}} \geq \underline{HA^{v,(1)\cap v,(2)}}$ and $A_0^{(v,(1)\cap v,(2))\prime} \in A_0^{v\prime,(1)}, A_0^{v\prime,(2)} \in X$.

Note that $A_0^{(v,(1)\cap v,(2))\prime}$ is a *null* hyperbox, iff $A^{v,(1)}$ and $A^{v,(2)}$ are not overlapped.

B. Characteristics of the Consequents of Fuzzy If-Then Rules

The characteristics of the consequent part of a fuzzy rule are explained, as follows.

Property 4. Characteristics of $y^v(\boldsymbol{x})$.

P4.1. A *fuzzy rule base* is *complete*, if $y^v(\boldsymbol{x})$, for all v and $\boldsymbol{x}$, is defined. Otherwise, the fuzzy rule base is *incomplete*.

P4.2. A fuzzy rule base is *monotone*, if $y^{v(r_{1,(1)},r_{2,(1)},\dots,r_{n,(1)})}(\boldsymbol{x}) \leq y^{v(r_{1,(2)},r_{2,(2)},\dots,r_{n,(2)})}(\boldsymbol{x})$ is true such that $(r_{1,(1)}, r_{2,(1)}, \dots, r_{n,(1)}) \leq (r_{1,(2)}, r_{2,(2)}, \dots, r_{n,(2)})$, for all $\boldsymbol{x}$. Otherwise, the fuzzy rule base is *non-monotone*.

P4.3. $y^v(\boldsymbol{x})$ is *monotone*, if $y^v\left(\boldsymbol{x}_{(1)} = (x_{1,(1)}, \dots, x_{i,(1)}, \dots, x_{n,(1)})\right) \leq y^v\left(\boldsymbol{x}_{(2)} = (x_{1,(2)}, \dots, x_{i,(2)}, \dots, x_{n,(2)})\right)$ for all $x_{i,(1)} \leq x_{i,(2)} \in X_i$, $i \in \{1, \dots, n\}$. Otherwise, $y^v(\boldsymbol{x})$ is *non-monotone*.

For clarity, a monotone fuzzy rule base (Property P4.2) and monotone $y^v(\boldsymbol{x})$ (Property P4.3) are listed as two properties. We define a restriction on each $y^v(\boldsymbol{x})$, subject to the support of A^v as in Definition 7. Property 5 is further presented.

C. Analysis of the TSK-FIS-product model

A restriction of $y^v(\boldsymbol{x})$ to $A_0^{v\prime}$ is presented in Definition 7, and its characteristics are in Property 5. With Definition 7, Theorem 1 is proposed.

Definition 7. The *restriction* of $y^v(\boldsymbol{x})$ to $A_0^{v\prime}$ is denoted as $y^v(\boldsymbol{x} | A_0^{v\prime})$.

Property 5. Characteristics of $y^v(\boldsymbol{x}|A_0^{v\prime})$.

P5.1. It there exist more than one $y^v(\boldsymbol{x})$, identical $y^v(\boldsymbol{x}|A_0^{v\prime})$ can be produced.

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Proof. Two non-identical $y^{v,(1)}(\boldsymbol{x})$ and $y^{v,(2)}(\boldsymbol{x})$ can produce an identical $y^{v}(\boldsymbol{x}|A_0^{v\prime})$.

P5.2. Non-monotone $y^{v}(\boldsymbol{x})$ (Property P4.3) can produce monotone $y^{v}(\boldsymbol{x}|A_0^{v\prime})$.

Proof. Even when $y^{v}(\boldsymbol{x})$ for all $\boldsymbol{x}$ is not monotone, its $y^{v}(\boldsymbol{x}|A_0^{v\prime})$ can be monotone, as its n-dimensional hyperbox (Definition 4) is a subset of X.

Theorem 1. A TSK-FIS-product model can be reduced to:
$y = f(\boldsymbol{x}) = \sum_{v=1}^{P} \overline{A^{v}(\boldsymbol{x})}\, y^{v}(\boldsymbol{x}) = \sum_{v=1}^{P} \overline{A^{v}(\boldsymbol{x})}\, y^{v}(\boldsymbol{x}|A_0^{v\prime})$

Proof. $y^{v}(\boldsymbol{x})$, which does not fall within $A_0^{v\prime}$, is not significant to $y = f(\boldsymbol{x})$, since $\overline{A^{v}(\boldsymbol{x})}$ is zero.

Theorem 1 indicates that the design of $y^{v}(\boldsymbol{x})$ for a TSK-FIS-product model can be reduced to the design of $y^{v}(\boldsymbol{x}|A_0^{v\prime})$. This is an important result for designing the parametric conditions of higher-order TSK-FIS-product models with functional consequents. Theorem 1, together with Properties 1, 3, 4, and 5, are important for analyzing the equivalence property (in the sense of [40]) associated with TSK-FIS-product models. In addition, $y^{v}(\boldsymbol{x}\,|A_0^{v\prime})$ of adjacent fuzzy if then rules can be used to obtain interpolated intermediate fuzzy if-then rules [41].

V. Expression of a Multi-input TSK-FIS-Product model as a series of single-input TSK-FIS-Product-Like models

A. Analysis

A TSK-FIS-product model, i.e., Eq. (3), is analyzed. Lemma 1 is presented.

Lemma 1. Given $s \in \{1,..,n\}\backslash\{i\}$, $\boldsymbol{x}_s$ is a vector with $n-1$ values, with x_i excluded; and $\boldsymbol{r}_s$ is a vector with $n-1$ integers, with r_i excluded. A multi-input TSK-FIS-product model, i.e., $f(\boldsymbol{x})$, can be expressed as follows.

$$f(\boldsymbol{x}) = f(x_i;\boldsymbol{x}_s) = \sum_{\forall \boldsymbol{r}_s}\left[\overline{\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)} \times y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x})\right] \quad (4)$$

where $\overline{\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)} = \frac{\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)}{\sum_{\forall \boldsymbol{r}_s}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)}$, $\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s) = \prod_{\forall s, s\neq i} A_s^{r_s}(x_s)$, and $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x}) = \frac{\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i) \times y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})}{\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)}$. There are $\prod_{\forall s, s\neq i} p_s$ potential combinations of $\overline{\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)}$, such that $\overline{\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)} \in \{\frac{0}{0}, [0,1]\}$.

Proof. A multi-input TSK-FIS-product model ($n > 1$), i.e., Eq. (3), can be expressed as follows;

$$f(\boldsymbol{x}) = f(x_i;\boldsymbol{x}_s) = \frac{\sum_{r_n=1}^{p_n}\cdots\sum_{r_2=1}^{p_2}\sum_{r_1=1}^{p_1} A_1^{r_1}(x_1)\times A_2^{r_2}(x_2)\times\dots\times A_n^{r_n}(x_n)\times y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})}{\sum_{r_n=1}^{p_n}\cdots\sum_{r_2=1}^{p_2}\sum_{r_1=1}^{p_1} A_1^{r_1}(x_1)\times A_2^{r_2}(x_2)\times\dots\times A_n^{r_n}} \quad (5)$$

$$f(x_i;\boldsymbol{x}_s) = \frac{\sum_{r_n=1}^{p_n}\cdots\sum_{r_2=1}^{p_2}\sum_{r_1=1}^{p_1} \varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\times A_i^{r_i}(x_i)\times y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})}{\sum_{r_n=1}^{p_n}\cdots\sum_{r_2=1}^{p_2}\sum_{r_1=1}^{p_1} \varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\times A_i^{r_i}(x_i)} \quad (6)$$

where $0 \le \varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s) \le 1$. From the denominator of Eq. (6),
$\sum_{r_n=1}^{p_n}\cdots\sum_{r_2=1}^{p_2}\sum_{r_1=1}^{p_1}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\times A_i^{r_i}(x_i) =$
$\sum_{r_{n\neq i}=1}^{p_n}\cdots\sum_{r_{2\neq i}=1}^{p_2}\sum_{r_{1\neq i}=1}^{p_1}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)$

From the numerator of Eq. (6),
$\sum_{r_n=1}^{p_n}\cdots\sum_{r_2=1}^{p_2}\sum_{r_1=1}^{p_1}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\times A_i^{r_i}(x_i)\times y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x}) =$
$\sum_{r_{n\neq i}=1}^{p_n}\cdots\sum_{r_{2\neq i}=1}^{p_2}\sum_{r_{1\neq i}=1}^{p_1}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)\, y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})$

Note that $f(\boldsymbol{x})$ can be written as follows.

$$f(\boldsymbol{x}) = \frac{\sum_{r_{n\neq i}=1}^{p_n}\cdots\sum_{r_{2\neq i}=1}^{p_2}\sum_{r_{1\neq i}=1}^{p_1}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)\, y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})}{\sum_{\forall \boldsymbol{r}_s}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)}$$

$$= \left[\frac{\sum_{r_{n\neq i}=1}^{p_n}\cdots\sum_{r_{2\neq i}=1}^{p_2}\sum_{r_{1\neq i}=1}^{p_1}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)}{\sum_{\forall \boldsymbol{r}_s}\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)}\frac{\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)\, y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})}{\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)}\right]$$

Equation (4) is a normalized weighted addition of $\overline{\varphi_{(\boldsymbol{r}_s)}(\boldsymbol{x}_s)}$ and $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x})$. The characteristics of $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x})$ are further outlined in Property 6.

Property 6. Characteristics of $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x})$

6.1. $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x})$ can be viewed as a single-input TSK-FIS-product-*like* model with fuzzy if-then rules in the form of $R_{i,(\boldsymbol{r}_s)}^{r_i}$: IF x_i is $A_i^{r_i}(x_i)$ THEN y is $y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})$.
6.2. The antecedent depends only on x_i.
6.3. The consequent depends on $\boldsymbol{x}$, whereby x_i is part of $\boldsymbol{x}$.
6.4. For each $\boldsymbol{r}_s$, p_i fuzzy if-then rules are expected.
6.5. $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x}) = \sum_{r_i=1}^{p_i}\overline{A_i^{r_i}(x_i)}\, y^{v}(\boldsymbol{x}|A_0^{v\prime})$, such that $\overline{A_i^{r_i}(x_i)} = \frac{A_i^{r_i}(x_i)}{\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)} \in \{\frac{0}{0}, [0,1]\}$. See the following proof.

Proof. From Lemma 1, $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x}) = \frac{\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)\times y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})}{\sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)}$,

$y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x}) = \sum_{r_i=1}^{p_i}\overline{A_i^{r_i}(x_i)}\, y^{v(r_1,r_2,\dots,r_n)}(\boldsymbol{x})$. From Theorem 1. $y^{v}(\boldsymbol{x})$ is restricted to $A_0^{v\prime}$. Thus, $y_{i,(\boldsymbol{r}_s)}(\boldsymbol{x}) = \sum_{r_i=1}^{p_i}\overline{A_i^{r_i}(x_i)}\, y^{v}(\boldsymbol{x}|A_0^{v\prime})$.

B. An Example

Consider a two-input zero-order TSK-FIS-product model (i.e., $n = 2$) with triangular and normal FMFs, as shown in Fig.3. Let $i = 1,2$ and $r_i = 1,2,3$. The three FMFs are designed such that for all $\boldsymbol{x} \in X \in \mathbb{R}^2$, $\sum_{v=1}^{P} A^{v}(\boldsymbol{x})$=0 does not occur, and $\mathbb{R}^2 = [0,1]^2$ is a bounded space. In addition, all $y^{v(r_1,r_2)}(\boldsymbol{x})$ are known.

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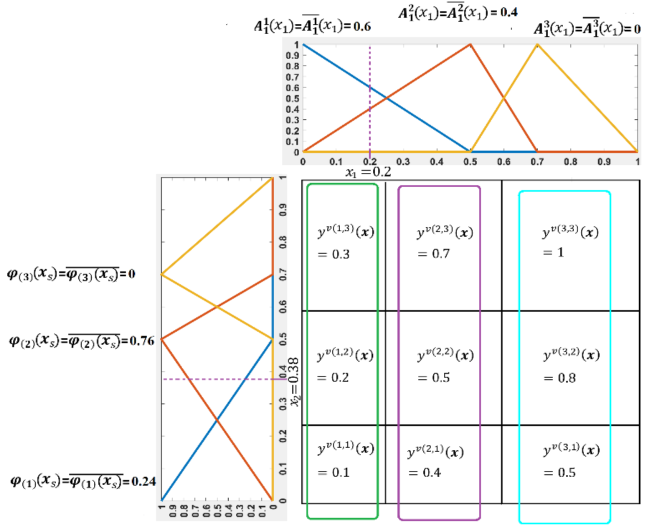

$f(\boldsymbol{x}) = \overline{\varphi_{(1)}(x_2)} \times y_{1,(r_s=(1))}(\boldsymbol{x}) + \overline{\varphi_{(2)}(x_2)} \times y_{1,(r_s=(2))}(\boldsymbol{x}) + \overline{\varphi_{(3)}(x_2)} \times y_{1,(r_s=(3))}(\boldsymbol{x}) = 0.368$

Fig.3. A two-input TSK-FIS-product model where $x_i = x_1$, $\boldsymbol{x}_s = (x_2)$. Green, purple, and turquoise rectangles are used to represent the single-input TSK-FIS-product-*like* models for $\boldsymbol{r}_2 = (1)$, $\boldsymbol{r}_2 = (2)$, and $\boldsymbol{r}_2 = (3)$, respectively

Given $\boldsymbol{x} = (x_1, x_2)$, we consider the explanation of $x_i = x_1$ and $\boldsymbol{x}_s = (x_2)$. Consider $x_i = x_1$, thus, $s \in \{2\}$, $\boldsymbol{x}_s = (x_2)$ is a vector with single value, with x_1 excluded; $\boldsymbol{r}_s = (r_2)$ is also a vector with single value. Three $\varphi_{(r_2)}(x_2)$ are expected, and $\varphi_{(r_2)}(x_2) = A_2^{r_2}(x_2)$. It is also possible to consider $x_i = x_2$ and $\boldsymbol{x}_s = (x_1)$, of course, with another expansion expected.

Let $\boldsymbol{x} = (x_1, x_2) = (0.2, 0.38)$. Following Eq. (6), the denominator of $f(0.2;\ 0.38)$ can be re-arranged to $\sum_{r_2=1}^{p_2=3} \varphi_{(r_2)}(x_2) \sum_{r_1=1}^{p_1} A_1^{r_1}(x_1) = (0.24 + 0.76 + 0)(0.6 + 0.4 + 0)$. The numerator of $f(0.2;\ 0.38)$ can be re-arranged to $\sum_{r_2=1}^{p_2=3} \sum_{r_1=1}^{p_1=3} \varphi_{(r_s)}(\boldsymbol{x}_s) \sum_{r_i=1}^{p_i} A_i^{r_i}(x_i)\, y^{v(r_1,r_2)} = 0.24(0.6 \times 0.1 + 0.4 \times 0.2 + 0 \times 0.3) + 0.76(0.6 \times 0.4 + 0.4 \times 0.5 + 0 \times 0.7) + 0(0.6 \times 0.5 + 0.4 \times 0.8 + 0 \times 1)$.

From *Lemma 1*, $\overline{\varphi_{(1)}} = \frac{0.24}{0.24+0.76+0}$, $\overline{\varphi_{(2)}} = \frac{0.76}{0.24+0.76+0}$, $\overline{\varphi_{(3)}} = \frac{0}{0.24+0.76+0}$. As such, three single-input TSK-FIS-product-*like* models can be obtained, as follows.

$y_{1,(r_s=(1))}(0.2,0.38) = \frac{(0.6\times y^{v(1,1)}(0.2,0.38)+0.4\times y^{v(1,2)}(0.2,0.38)+0\times y^{v(1,3)}(0.2,0.38))}{(0.6+0.4+0)}$;

$y_{1,(r_s=(2))}(0.2,0.38) = \frac{(0.6\times y^{v(2,1)}(0.2,0.38)+0.4\times y^{v(2,2)}(0.2,0.38)+0\times y^{v(2,3)}(0.2,0.38))}{(0.6+0.4+0)}$;

$y_{1,(r_s=(3))}(0.2,0.38) = \frac{(0.6\times y^{v(3,1)}(0.2,0.38)+0.4\times y^{v(3,2)}(0.2,0.38)+0\times y^{v(3,3)}(0.2,0.38))}{(0.6+0.4+0)}$.

There are $p_i = 3$ fuzzy if-then rules in the form of IF x_1 is $A_1^{r_1}(x_1)$ THEN y is $y^{v(r_1,r_2)}(\boldsymbol{x})$, for each r_s. The antecedent depends only on x_1. The consequent, i.e., $y^{v(r_1,r_2)}(\boldsymbol{x})$, depends on two variables, i.e., x_1 and x_2. Note that $\varphi_{(r_s)}(x_2)$ only depends on x_2. The expansion of the entire TSK-FIS-product model depends on both x_1 and x_2 (see Fig. 3).

In Fig. 3, each $y_{1,(r_s)}(\boldsymbol{x})$ is represented as a rectangle for $\boldsymbol{r}_s = (1)$, $\boldsymbol{r}_s = (2)$, and $\boldsymbol{r}_s = (3)$, each with a different color, respectively. Each rectangle represents a group of three single-input fuzzy if-then rules, where $r_1 = \{1,2,3\}$. From Eq. (4), $f(\boldsymbol{x}) = \overline{\varphi_{(1)}(x_2)} \times y_{1,(r_s=(1))}(\boldsymbol{x}) + \overline{\varphi_{(2)}(x_2)} \times y_{1,(r_s=(2))}(\boldsymbol{x}) + \overline{\varphi_{(3)}(x_2)} \times y_{1,(r_s=(3))}(\boldsymbol{x}) = 0.368$. Notice that when all FMFs are normal with a strong partition [17], $\varphi_{(r_s)}(\boldsymbol{x}_s) = \overline{\varphi_{(r_s)}(\boldsymbol{x}_s)} \in [0,1]$ and $A_1^{r_1}(x_1) = \overline{A_1^{r_1}(x_1)}$.

C. Remarks

In [42], a discussion on the use of fuzzy rules pertaining to single-input FIS models for constructing a multi-input FIS model. In this study, such expression is exploited to analyze the monotone property of a TSK-FIS-product model. Note that it is not the aim of this study to validate the argument that multi-input fuzzy if-then rules from human experts can be gathered through single-input fuzzy if-then rules with weights imposed.

In the current literature, there are a number of *fuzzy basis function* or expansion for TSK-FIS models (see [43] [44]). In this study, we consider an additional possibility for $A^v(\boldsymbol{x})$ and $\overline{A^v(\boldsymbol{x})}$, whereby $\overline{A^v(\boldsymbol{x})}$ could be $\frac{0}{0}$ in our expansion. This expansion is new and important for designing sparse and monotone TSK-FIS-product models (see [5][16]).

VI. Monotone TSK-FIS-Product models

A definition for a TSK-FIS-product model to be of monotone and non-decreasing is presented in Definition 8. In this section, several theorems and remarks pertaining to the necessary conditions and sufficient conditions for the TSK-FIS-product model to be monotone are elaborated.

Definition 8. A TSK-FIS-product model (denoted as f) is a monotone (non-decreasing) FIS iff it is a mapping $f: X \to Y$ that satisfies $f\left(\boldsymbol{x}_{(1)} = (x_{1,(1)}, \dots, x_{i,(1)}, \dots, x_{n,(1)})\right) \le f\left(\boldsymbol{x}_{(2)} = (x_{1,(2)}, \dots, x_{i,(2)}, \dots, x_{n,(2)})\right)$ for all $x_{i,(1)} \le x_{i,(2)} \in X_i$, $i \in \{1, \dots, n\}$, with an n-dimensional input space $X \in \mathbb{R}^n$ and an output space $Y \in \mathbb{R}$.

A. The necessary condition for a monotone TSK-FIS-product model

Based on the *necessary condition* and *sufficient condition* in [45], the following assumptions are made. A mathematical condition for a TSK-FIS-product model to be monotone is denoted as *necessary* when it is impossible for the TSK-FIS-product model to be monotone without the mathematical condition. If a counter example for the TSK-FIS-product model to be monotone without the mathematical condition exists, then the said mathematical condition is not necessary. On the other hand, a mathematical condition for a TSK-FIS-product model to be monotone is denoted as *sufficient* when the mathematical condition guarantees the TSK-FIS-product model to be monotone. Note that a TSK-FIS-product model can be monotone without the sufficient condition.

Theorem 2. $\sum_{v=1}^{P} A^v(\boldsymbol{x}) > 0$, for all $\boldsymbol{x}$, is a necessary condition for a TSK-FIS-product model to be monotone.

Proof. Property P1.2. If $\sum_{v=1}^{P} A^v(\boldsymbol{x}) = 0$, $f(\boldsymbol{x})$ is undetermined.

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Theorem 3. All P $y^v(\boldsymbol{x}|A_0^{v\prime})$ entities are known, which constitute a necessary condition. However, $y^v(\boldsymbol{x})$ may not be known for the entire $\boldsymbol{x}$.

Proof. The computation of a TSK-FIS-product model for all $\boldsymbol{x}$, is impossible if any of $y^v(\boldsymbol{x}|A_0^{v\prime})$ is unknown. From Theorem 1, it is not necessary to define $y^v(\boldsymbol{x})$ for the entire $\boldsymbol{x}$, in order to perform computation of the TSK-FIS-product model.

Theorems 2 and 3 are a joint necessary condition for the TSK-FIS-product model (Eq. (3)). Remarks 2-4 are presented, whereby the restriction of $\sum_{v=1}^{P} A^v(\boldsymbol{x}) > 0$, for all $\boldsymbol{x}$ (Theorem 2) is satisfied; and $y^v(\boldsymbol{x}|A_0^{v\prime})$ are known for all $\boldsymbol{x}$ (Theorem 3).

Remark 2. The specification of $A^v(\boldsymbol{x})$ *alone* is not a necessary condition for designing a monotone TSK-FIS-product model.

Proof. If all $y^v(\boldsymbol{x})$ are identical for all v, $y = f(\boldsymbol{x}) = y^v(\boldsymbol{x})$, is always true. If there exists a counter example that the TSK-FIS-product model is monotone regardless of the specification of $A^v(\boldsymbol{x})$, then the specification of $A^v(\boldsymbol{x})$ alone is not a necessary condition.

Remark 3. Monotone $y^v(\boldsymbol{x})$ (Property P4.3) *alone* is not a necessary condition for designing a monotone TSK-FIS-product model.

Proof. Deduced from Theorem 1.

Remark 4. A monotone fuzzy rule base (Property P4.2) *alone* is not a necessary condition for designing a monotone TSK-FIS-product model.

Proof. Consider $\mathcal{P}(\boldsymbol{x}) = \left(\overline{A^1(\boldsymbol{x})}, \overline{A^2(\boldsymbol{x})}, \dots, \overline{A^P(\boldsymbol{x})}\right)$ remains identical for all $\boldsymbol{x}$, i.e., orness of $\mathcal{P}(\boldsymbol{x})$ is *static* when $\boldsymbol{x}$ increases. Given a zero-order TSK-FIS-product model, $y^v(\boldsymbol{x})$ is a numeral, both $A^v(\boldsymbol{x})$ and $y^v(\boldsymbol{x})$, for $v = 1,2,\dots,P$, are no longer a function of $\boldsymbol{x}$. From Eq. (3), the TSK-FIS-product model can be reduced to $f(\boldsymbol{x}) = \sum_{v=1}^{P} \overline{A^v}\, y^v$. In this case, $f(\boldsymbol{x})$ always produces a constant numeral satisfying Definition 8, even when Property P4.2 does not stand. If there exists a counter example that the TSK-FIS-product model can be monotone with a non-monotone fuzzy rule base, then a monotone fuzzy rule base, alone, is not a necessary condition.

Theorem 2 indicates that $\sum_{v=1}^{P} A^v(\boldsymbol{x}) > 0$ is a necessary condition for constructing a monotone TSK-FIS-product model. Theorem 3 specifies that the fuzzy rules should be complete, in the sense that all $y^v(\boldsymbol{x}|A_0^{v\prime})$ are defined. This is more general and less restrictive than all $y^v(\boldsymbol{x})$ need to be known for the monotone TSK-FIS-product model. Both Theorems 2 and 3 signify the importance of (monotone) fuzzy rule interpolation [16][31][32] and interval methods (e.g., Monotone interval FIS models [15]) for constructing a monotone TSK-FIS-product model. From Theorems 2 and 3, we can deduce Remark 5.

Remark 5.

R5.1. Given a monotone TSK-FIS-product model, $\sum_{v=1}^{P} A^v(\boldsymbol{x}) > 0$ is always true.

R5.2. Given a monotone TSK-FIS-product model, all $y^v(\boldsymbol{x}|A_0^{v\prime})$ are known.

In addition, Remarks 2 to 4 indicate that based on each specification of $A^v(\boldsymbol{x})$, a monotone $y^v(\boldsymbol{x})$ or monotone fuzzy rule base by itself alone is not a necessary condition for designing a monotone TSK-FIS-product model.

B. The sufficient condition for a monotone TSK-FIS-product model

Inspired by increasing *orness* in OWA and the *hyperboxe* structure, a joint *sufficient condition* of a TSK-FIS-product model to be monotone is derived, as in Proposition 1. It considers FMFs with single and continuous support, as well as an additional condition for two overlapped FMFs for all X_i.

Proposition 1. A set of sufficient conditions for a TSK-FIS-product model to be monotone (Definition 8), subject to a restriction on the maximal of two FMFs overlap each other for all X_i, is presented, as follows.

1.1. From Theorem 2, $\sum_{v=1}^{P} A^v(\boldsymbol{x}) > 0$, for all $\boldsymbol{x}$, is satisfied.

1.2. From Theorem 3, at the consequent part, all $y^v(\boldsymbol{x}|A_0^{v\prime})$ are known, but $y^v(\boldsymbol{x})$ may not be known for the entire $\boldsymbol{x}$.

1.3. At the antecedent part, all $A_i^{r_i}(x_i)$ are designed such that $\left(0,0,..,\overline{A_i^{r_i}(x_{i,(2)})}, \overline{A_i^{r_i+1}(x_{i,(2)})}, \dots,0,0\right) = \left(0,0,..,\overline{A_i^{r_i}(x_{i,(1)})} - \epsilon, \overline{A_i^{r_i+1}(x_{i,(1)})} + \epsilon, \dots,0,0\right)$, where $\epsilon \geq 0$; is always true, $\forall x_{i,(2)} > x_{i,(1)}$, $\forall i$, $\forall x_i$, and $r_i \in \{1,2,3,\dots,p_i - 1\}$.

1.4. All $y^v(\boldsymbol{x}|A_0^{v\prime})$ are monotone, i.e., $y^v(\boldsymbol{x}_{(1)} = (x_{1,(1)},\dots,x_{i,(1)},\dots,x_{n,(1)})|A_0^{v\prime}) \leq y^v(\boldsymbol{x}_{(2)} = (x_{1,(2)},\dots,x_{i,(2)},\dots,x_{n,(2)})|A_0^{v\prime})$ for all $x_{i,(1)} \leq x_{i,(2)} \in X_i$, and all v, $i \in \{1,\dots,n\}$, but $y^v(\boldsymbol{x})$ may not be monotone entirely (see Property P5.2).

1.5. $y^{v(r_{1,(1)},r_{2,(1)},\dots,r_{n,(1)})}(\boldsymbol{x}) \leq y^{v(r_{1,(2)},r_{2,(2)},\dots,r_{n,(2)})}(\boldsymbol{x})$, for all $\boldsymbol{x} \in A_0^{(v,(1)\cap v,(2))\prime}$, and $(r_{1,(1)},r_{2,(1)},\dots,r_{n,(1)}) \leq (r_{1,(2)},r_{2,(2)},\dots,r_{n,(2)})$.

Proof. From Definition 2, a TSK-FIS-product model can be expressed as P-ary normalized weighted average of $\overline{A^v(\boldsymbol{x})}$ and $y^v(\boldsymbol{x})$. In line with Proposition 1.1, $\sum_{v=1}^{P} \overline{A^v(\boldsymbol{x})} = 1$ is always true. In line with Proposition 1.2, $y^v(\boldsymbol{x}|A_0^{v\prime})$ can always be determined and known. Both are the necessary conditions to be satisfied (see Theorems 2 and 3), ensuring a valid output from a monotone TSK-FIS-product model.

Propositions 1.4 and 1.5 ensure that $y^{v(r_{1,(1)},r_{2,(1)},\dots,r_{n,(1)})}(\boldsymbol{x}) \leq y^{v(r_{1,(2)},r_{2,(2)},\dots,r_{n,(2)})}(\boldsymbol{x})$ is always true, for each $\boldsymbol{x}_{(1)} \leq \boldsymbol{x}_{(2)}$ and $(r_{1,(1)},r_{2,(1)},\dots,r_{n,(1)}) \leq (r_{1,(2)},r_{2,(2)},\dots,r_{n,(2)})$. Deduced from the OWA principle [23], along with Proposition 1.2, the TSK-FIS-product model is always monotone.

C. Discussion

Proposition 1 also leads to Remark 6, as follows. Remark 6 denotes that the specification of p_i is not a necessary condition. Remarks 7 further explains why Proposition 1 is more general.

Remark 6.

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R6.1. A monotone single-input/multi-input TSK-FIS-product model can be obtained with FMFs having $p_i = 1$, for all *i.*

Proof. If $p_i = 1$, for all i, $\left(\overline{A_i^1(x_i)}\right) = (1)$ for all x_i is always true. Orness of $\left(\overline{A_i^1(x_i)}\right)$ remains static when x_i increases. Proposition 1 is satisfied.

R6.2. A monotone TSK-FIS-product model can be obtained with an arbitrary number of FMFs. Propositions 1.1 and 1.3 can be achieved with an arbitrary number of FMFs.

R6.3. A monotone TSK-FIS-product model can be obtained with an arbitrary number of fuzzy if-then rules.

Remark 7 Proposition 1.3 indicates more general conditions for designing two overlapped FMFs.

Proof. We consider FMFs with single and continuous support, which is more general (see Remark 1), i.e., relaxing the requirements of convexity, having no jump discontinuity, and normality in designing FMFs. See Section VII (C) for illustrative examples.

Remark 8 Proposition 1.4 is more general.

Proof. We allow $y^v(\boldsymbol{x})$ to be non-monotone. See Section VII (A) for an illustrative example.

Our analysis also indicates that research on the necessary conditions for designing a monotone TSK-FIS-product model [2] should focus on specific cases with meaningful restrictions, e.g., for the TSK-FIS-product model to be *explainable* [46][47], or for the TSK-FIS-product model to have all non-identical consequents.

VII. Case Studies

A. A monotone TSK-FIS-product Occurrence Model in FMEA [37]

In FMEA, obtaining the ratings of severity, occurrence and detection scores is a tedious and time-consuming task. Here, we design a monotone TSK-FIS-product occurrence model [37] to automate the occurrence score ratings, i.e., $y = f(x_1)$. Based on the information from an extended occurrence scale table from a real semiconductor manufacturing facility [21], as presented in Table I, the developed occurrence model is useful for *dynamic risk evaluation,* as indicated in [48].

TABLE I
SCALE TABLE FOR OCCURRENCE SCORE [37]

Occurrence score	Linguistic terms used to describe y^v	Linguistic terms used to describe x_1, i.e., $(A_1^{r_1})$	Average number of failure occurred in 52 weeks (x_1)	
			Lower limit	Upper limit
10~9	Very high (y^6)	Many/shift, many/day (A_1^6)	301	1000
8~7	High (y^5)	Many/week, few/week (A_1^5)	53	300
6~4	Moderate (y^4)	Once/week, several/month (A_1^4)	13	52
3	Low (y^3)	Once/month (A_1^3)	5	12
2	Very low (y^2)	Once/quarter (A_1^2)	3	4
1	Remote (y^1)	Once ever (A_1^1)	1	2

Consider a monitoring period of 52 weeks, as determined by FMEA users. An input $x_1 \in [1,1000]$, which can be in the form of a logarithmic scale of $x_{1,log} \in [0,3]$, represents the *average number of failures occurred* within a monitoring period. The output occurrence score is $y \in [1,10]$. Note that the logarithmic scale is often used when the input domain is large [37]. It is difficult to design a monotone TSK-FIS-product model for the input domain. As an example, referring to Table I, the first FMF covers interval [1, 2] while the last FMF covers [301, 1000], both are of x_1.

Note that FMEA users expect that if x_1 increases, y should increase too, i.e., a monotone relationship needs to be obeyed. As such, it is vital for the occurrence score to preserve such a monotone relationship in order to accurately deduce the risk priority number in FMEA. This further ensures valid and meaningful comparisons among different potential failure modes in FMEA.

Table I can be viewed as a mapping from "average number of failures occurred/52 weeks" to "occurrence score". The information can be explained using six monotone fuzzy if-then rules, as presented in Fig. 4. As an example, the highlighted row in Fig. 4 corresponds to "IF x_1(or $x_{1,log}$) is *Many/day* THEN y is *Remote*". For performance comparison, fuzzy if-then rules with different functional consequents, i.e., zero-order, first-order, and second-order consequents, are presented in Fig. 4. Note that the linguistic term "*Remote*" for y is represented by a numeral value of $y^1 = 1$.

Figures 5 and 6 depict the FMFs, subject to the restriction that the maximal of two FMFs overlap each other, for linguistic terms x_1 and $x_{1,log}$, respectively. Notice that Propositions 1.1 and 1.3 are satisfied. The consequents of all fuzzy rules are designed such that Propositions 1.2, 1.4, and 1.5 are satisfied. As an example, for the zero-order case, the linguistic terms of Remote, Very low, Low, Moderate, High, and Very high, are represented by the numerical output $y^v(\boldsymbol{x}) \in \{1,2,3,5,7.5,10\}$, respectively. All consequents are known, and $y^1 < y^2 < y^3 < y^4 < y^5 < y^6$, therefore Propositions 1.2, 1.4, and 1.5 are satisfied. Following Proposition 1, a monotone f is expected. This example shows the possibility of obtaining a monotone TSK-FIS-product model with non-monotone functions, i.e., quadratic functions, as functional consequents. Notice that the obtained second-order TSK-FIS-product occurrence model can be constructed with the monotone restricted quadratic functions.

	Fuzzy if-then rules				Zero-order consequent	First-order consequent	Second-order consequent
R^1	if x_1 is	Many/day	then y is	Remote	$y^1 = 1.000$	$y^1 = 1.000$	$y^1 = 1.000$
R^2		Many/week		Very Low	$y^2 = 2.000$	$y^2 = 2.000$	$y^2 = 2.000$
R^3		Once/week		Low	$y^3 = 3.000$	$y^3 = 3.000$	$y^3 = 3.000$
R^4		Once/month		Moderate	$y^4 = 5.000$	$y^4 = 0.011x_1 + 6.48$	$y^4 = 1.74\text{e-}5(x_1)^2 + 5.85$
R^5		Once/quarter		High	$y^5 = 7.500$	$y^5 = 0.002x_1 + 7.94$	$y^5 = 3.32\text{e-}6(x_1)^2 + 7.31$
R^6		Once ever		Very High	$y^6 = 10.000$	$y^6 = 0.001x_1 + 8.67$	$y^6 = 2.00\text{e-}6(x_1)^2 + 8$

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Fig. 4. Fuzzy if-then rules of the zero-order [37], first-order and second-order TSK-FIS-product occurrence models (all numerals are rounded to three decimal places)

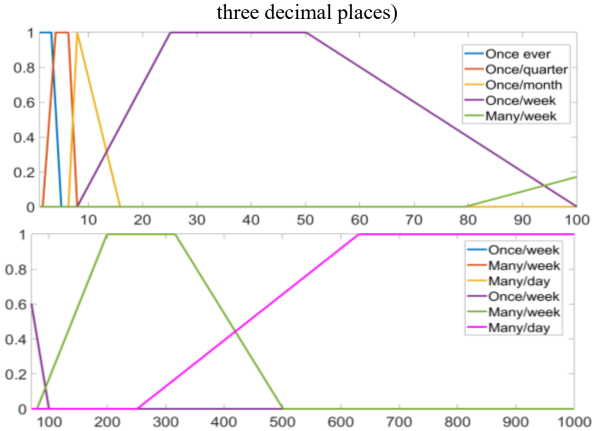
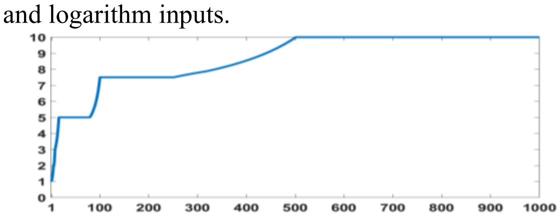


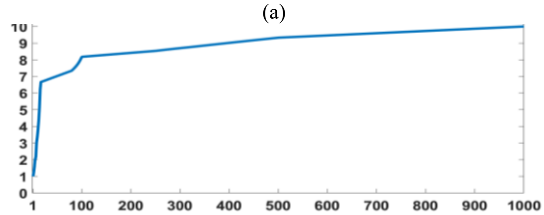
Fig. 5. Design of FMFs for the TSK-FIS-product occurrence model
Fig. 6. Design of FMFs for the TSK-FIS-product occurrence model on a logarithmic scale [37]

Figs 7(a), 7(b) and 7(c) depict the plot of $y = f(x_1)$ for the TSK-FIS-product occurrence model with zero-order, first-order and second-order consequents, respectively. All models are monotone, and all meet the boundary conditions. With zero-order consequents, it can be observed that the line remains static, i.e., $y = 10$, after x_1>500. However, a gradual increase can be obtained in the case of the first-order and second-order consequents. This indicates the ability to represent the increasing risk of the occurrence score with respect to average number of failures occurred/52 weeks. A plot of $y = f(x_{1,log})$, based on the zero-order TSK-FIS-product occurrence model, is depicted in Fig. 8. It can be clearly observed that the zero-order TSK-FIS-product occurrence model is monotone, and it satisfies the boundary conditions.

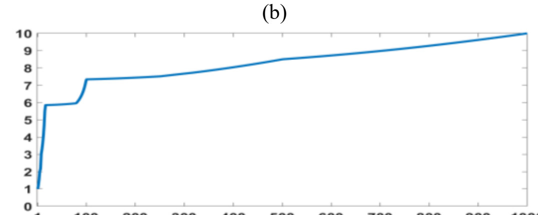
In summary, with our formulated propositions, a monotone TSK-FIS-product occurrence model can be designed, with both real and logarithm inputs.



(a)



(b)



(c)

Fig. 7. Surface plot of the TSK-FIS-product occurrence model with (a) zero-order [37], (b) first-order and, (c) second-order consequents

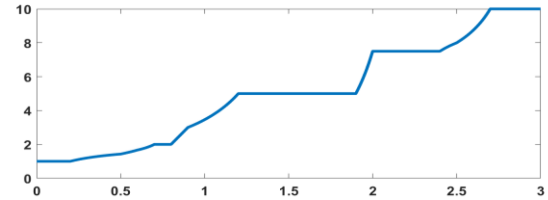
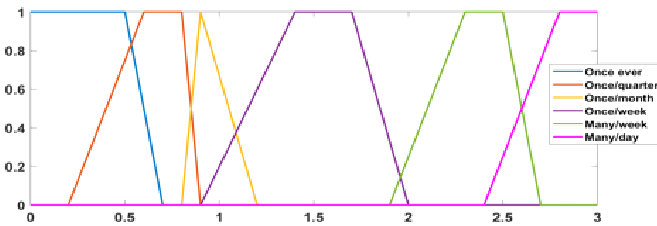


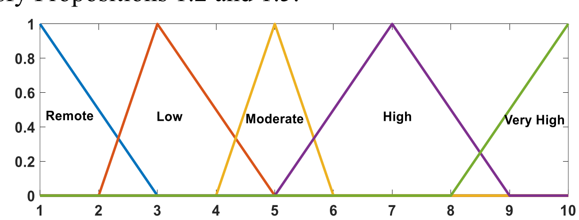
Fig. 8. Surface plot of the zero-order TSK-FIS-product occurrence model with a logarithmic scale

B. A monotone TSK-FIS-product Risk Priority Number Model [49][50]

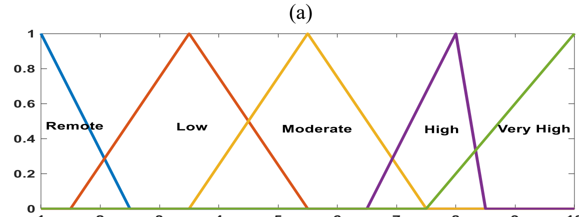
A set of benchmark fuzzy rules related to a Risk Priority Number (RPN) model in FMEA for a sewage treatment plant [49][50] is considered. The RPN model considers severity



(x_1), occurrence (x_2), and detection (x_3) scores, as the input risk factors, i.e., $\boldsymbol{x} = (x_1, x_2, x_3) \in [1,10]$. They produce a fuzzy RPN score (f_{RPN}), as the output. The importance of the monotone property of an RPN model has been highlighted in [14][16][21]. In this section, we design a monotone TSK-FIS-product RPN model with FMFs and fuzzy rule base provided in [49][50]. The provided FMFs design are presented in Fig. 9, which satisfy Propositions 1.1 and 1.3. The fuzzy RPN score is associated with linguistic terms of Low, Fairly low, Moderate, Fairly high, and High, which are represented by the numerical output $y^v(\boldsymbol{x}) \in \{1,2.5,3.7,6.1,10\}$. In short, $i = 1,2,3$ and $r_i = 1,2,3,4,5$. All $y^{v(r_1,r_2,r_3)}(\boldsymbol{x})$ are numerals and known, which satisfy Propositions 1.2 and 1.5.



(a)



(b)

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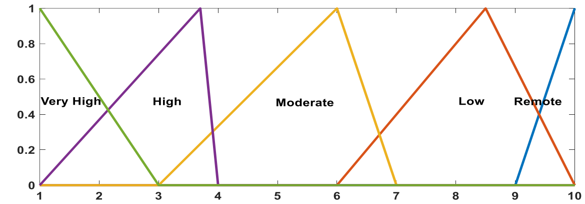
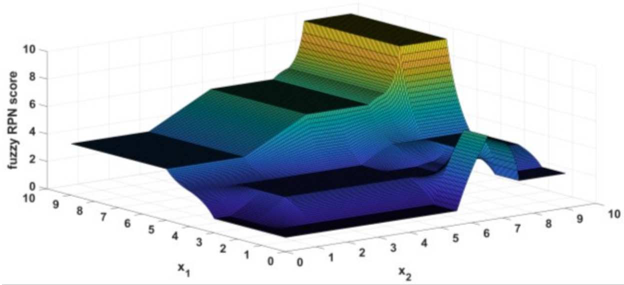


(c)

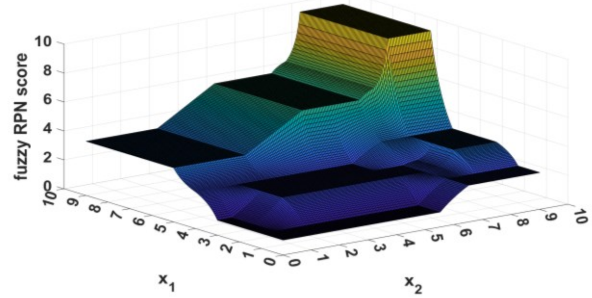
Fig. 9. Design of FMFs for inputs (a) Severity (x_1), (b) occurrence (x_2), and (c) Detection (x_3) [49] [50]

However, from [49][50], the provided fuzzy rules are non-monotone (also see [14]), i.e., Proposition 1.4 is violated. As an example, this violation can be observed from the provided fuzzy rules $R^{v(1,4,2)}$ and $R^{v(1,5,2)}$ with severity, occurrence and detection of Remote, High, and High, as well as Remote, Very High, and High, respectively. $R^{v(1,4,2)}$ and $R^{v(1,5,2)}$ are associated with the consequents of Fairly High and Fairly Low, respectively. While the consequent $y^{v(1,5,2)}$ is expected to be higher than or equal to $y^{v(1,4,2)}$, this is not the case. As an example, $(f_{RPN}(1,8,3) = 5.5) > (f_{RPN}(3,8,3) = 2.7)$ is obtained using Eq. (3). This drawback can be rectified using the approach in [15], which considers f_{RPN} as an interval, i.e., $f_{RPN}(\boldsymbol{x}) = \left[\underline{f_{RPN}(\boldsymbol{x})}, \overline{f_{RPN}(\boldsymbol{x})}\right]$. All $y^{v(r_1,r_2,r_3)}(\boldsymbol{x})$ are intervals too, i.e., $\left[\underline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})}, \overline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})}\right]$, such that $\underline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})} = \min\left(\min_{\forall A^{v(t_1\ge r_1,t_2\ge r_2,t_3\ge r_3)}}[y^v(\boldsymbol{x}),10], \max_{\forall A^{v(t_1\le r_1,t_2\le r_2,t_3\le r_3)}}[y^v(\boldsymbol{x}),1]\right)$, $\overline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})} = \max\left(\min_{\forall A^{v(t_1\ge r_1,t_2\ge r_2,t_3\ge r_3)}}[y^v(\boldsymbol{x}),10], \max_{\forall A^{v(t_1\le r_1,t_2\le r_2,t_3\le r_3)}}[y^v(\boldsymbol{x}),1]\right)$, where $t_1 \in \{1,\dots,p_1\}, t_2 \in \{1,\dots,p_2\}, t_3 \in \{1,\dots,p_3\}$ is an integer.

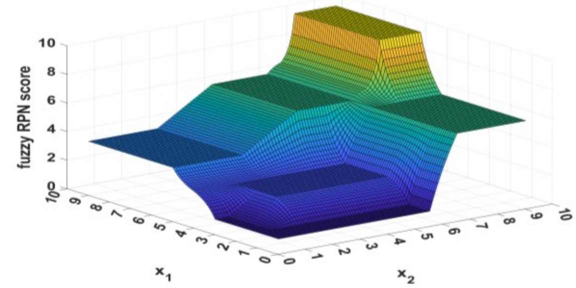
Proposition 1.4 is satisfied for $\underline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})}, \overline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})}$. With a monotone TSK-FIS-product model, $\underline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})}$ and $\overline{y^{v(r_1,r_2,r_3)}(\boldsymbol{x})}$ are used to yield $\underline{f_{RPN}(\boldsymbol{x})}$ and $\overline{f_{RPN}(\boldsymbol{x})}$, respectively. The mean value, $f_{RPN,mean}(\boldsymbol{x}) = 0.5 \times \left(\underline{f_{RPN}(\boldsymbol{x})} + \overline{f_{RPN}(\boldsymbol{x})}\right)$, is considered With the same example, $(f_{RPN}(1,8,3) = [2,6]) < (f_{RPN}(3,8,3) = [3,6])$ and $(f_{RPN,mean}(1,8,3) = 4) = (f_{RPN,mean}(3,8,3) = 4)$ can be obtained, satisfying the monotone property. The surface plot of f_{RPN} versus x_1 and x_2, with x_3=3, using Eq. (3), is depicted in Fig 10(a). A non-monotone surface plot is obtained with the original fuzzy rule base from [49] [50], $f_{RPN}(\boldsymbol{x})$. The surface plots of $\underline{f_{RPN}(\boldsymbol{x})}$, $\overline{f_{RPN}(\boldsymbol{x})}$ and $f_{RPN,mean}$, versus x_1 and x_2, are depicted in Figs. 10 (b), (c), and (d). Monotone surface plots are obtained.



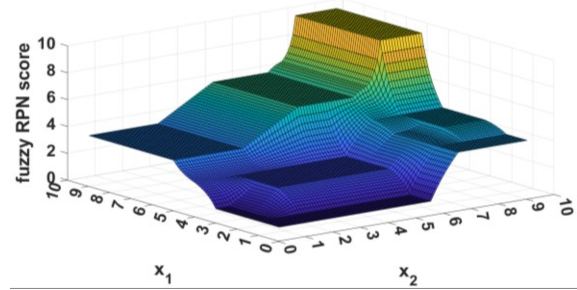
(a)



(b)



(c)



(d)

Fig. 10. Surface plots of the RPN score versus severity and occurrence, with detection=3, using (a) $f_{RPN}(\boldsymbol{x})$, (b). $\underline{f_{RPN}(\boldsymbol{x})}$, (c) $\overline{f_{RPN}(\boldsymbol{x})}$ and, (d) $f_{RPN,mean}(\boldsymbol{x})$

C. A monotone TSK-FIS-product model for image edge detection [38]

In [38], a four-phase geometric methodology was proposed for edge detection in image processing, namely (1) conditioning, (2) feature extraction, (3) blending, and (4) scaling. According to [38], a blending function that is nearly concave in the neighborhood of the feature space origin is denoted as a *waterfall function*. One fundamental property of the waterfall function is its monotone nature (see Fig. 8 in [38]).

This section examines the use of Proposition 1 to design a monotone TSK-FIS-product model as a blending function for image edge detection. Based on [38], two inputs, i.e., $\boldsymbol{x} = (x_1, x_2) \in [0,4]$, constitute the horizontal and vertical Sobel components in the feature extraction phase. The FMFs for x_1 and x_2 are presented in Fig. 11, which satisfy Propositions 1.1 and 1.3. The output of the blending function, i.e., u, is obtained using Eq. (3). Four fuzzy rules with four parameters, i.e., τ, χ, γ, and ω, are formed, as follows [38].

Rule 1: If $x_1 = L$ and $x_2 = L$ *Then* $u_1(\boldsymbol{x}) = {x_1}^{\tau} + {x_2}^{\tau}$ (7.a)

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Rule 2: If $x_1 = L$ and $x_2 = H$ *Then* $u_2(\boldsymbol{x}) = \chi$ (7.b)
Rule 3: If $x_1 = H$ and $x_2 = L$ *Then* $u_3(\boldsymbol{x}) = \gamma$ (7.c)
Rule 4: If $x_1 = H$ and $x_2 = H$ *Then* $u_4(\boldsymbol{x}) = \omega$ (7.d)

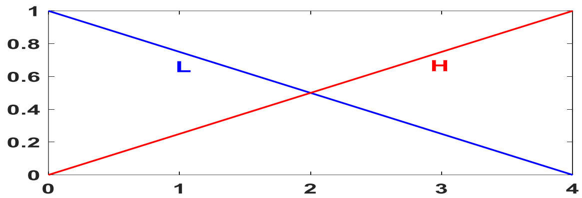
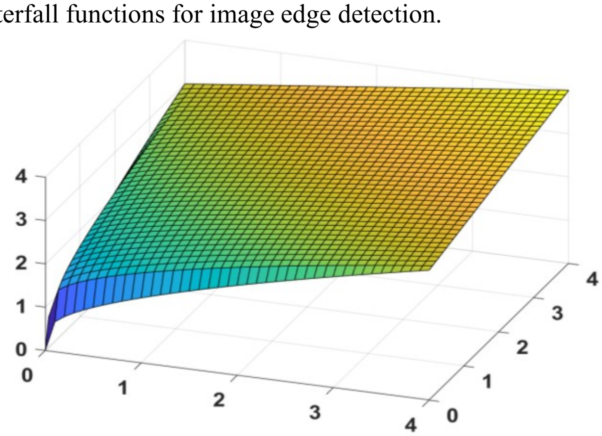


Fig. 11. Design of FMFs for inputs x_1 and x_2 [38]

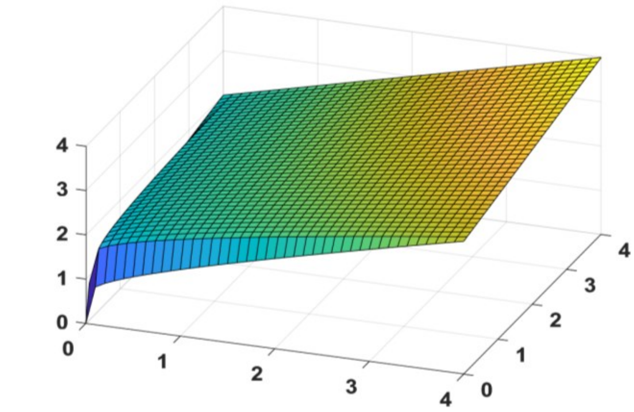
To satisfy Propositions 1.2, 1.4 and 1.5, a constraint with respect to τ, χ, γ, and ω is introduced, i.e.,
constraint: subject to $2 \times 4^{\tau} \leq \chi, \gamma \leq \omega$ (8)

Then, a *feasible solution* for the constraint in Eq. (8) can be obtained. Considering the constraint in Eq. (8), two waterfall functions based on monotone TSK-FIS-product models are established: (a) $\tau = 0.2$, $\chi = 3$, $\gamma = 3$, $\omega = 4$, and (b) $\tau = 0.1$, $\chi = 2$, $\gamma = 3$, $\omega = 4$. As depicted in Fig. 12, both surface plots are roughly concave near the feature space origin, with a monotone response far from the origin.

Three popular images of size 512×512 are employed for evaluation, i.e., cameraman, peppers, and mandrill, as shown in Fig 13(a). A non-monotone waterfall function based on the description in [38] is first derived, and the resulting images are depicted in Fig 13 (b). Comparatively, Figs. 13(c) and 13(d) depict the generated images based on the two formulated blending functions in Figs. 12(a) and 12(b), respectively. It can be clearly visualized that both waterfall functions constructed with monotone TSK-FIS-product models can produce geometrical characteristics and detailed structures within the images. These results indicate the usefulness of Proposition 1 in constructing monotone TSK-FIS-product models as effective waterfall functions for image edge detection.

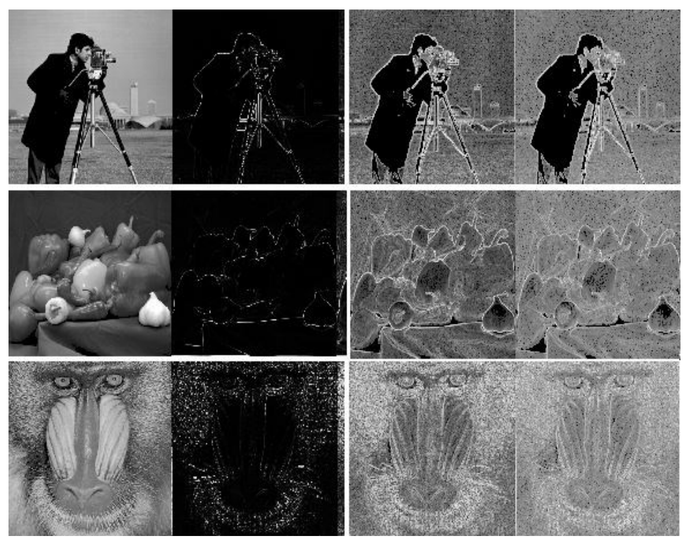


(a)



(b)

Fig. 12. Surface plot of monotone TSK-FIS-product models as waterfall functions with different parameters: (a) $\tau = 0.2$, $\chi = 3$, $\gamma = 3$, $\omega = 4$; (b) $\tau = 0.1$, $\chi = 2$, $\gamma = 3$, $\omega = 4$



(a) (b) (c) (d)

Fig. 13 (a) original images (b) images from a non-monotone waterfall function; (c) images from the monotone waterfall function shown in Fig. 12(a); (d) images from the monotone waterfall function shown in Fig. 12(b)

D. Discussion

In general, two challenges in constructing monotone TSK-FIS-product models are the design of FMFs and fuzzy if-then rules. Figures 14 consists of three FMF designs for a monotone TSK-FIS-product occurrence model, each adopts a different FMF design with single and continuous support (see Definition 3). Each FMF design covers a mixture of FMFs comprising convex and non-convex, normal and sub-normal, and with jump discontinuous characteristics. In accordance with Property P3.2, different FMF designs can yield the same $\mathcal{P}(x_1)$, for all x_1, based on the same monotone TSK-FIS-product model. As an example, instead of the FMF design in Fig. 6, identical $\mathcal{P}(x_1)$, for all x_1, can be obtained using the FMF designs shown in Figs. 14(a), 14(b), and 14(c) too.

When $x_1 = 0.1$, all FMF designs in Figs. 6 and 14 produce the same $\mathcal{P}(0.1)$, i.e., $\mathcal{P}(0.1) = (1,0,0,0,0,0)^T$. As such, an identical $f(x_1)$ can be obtained with different FMF designs and

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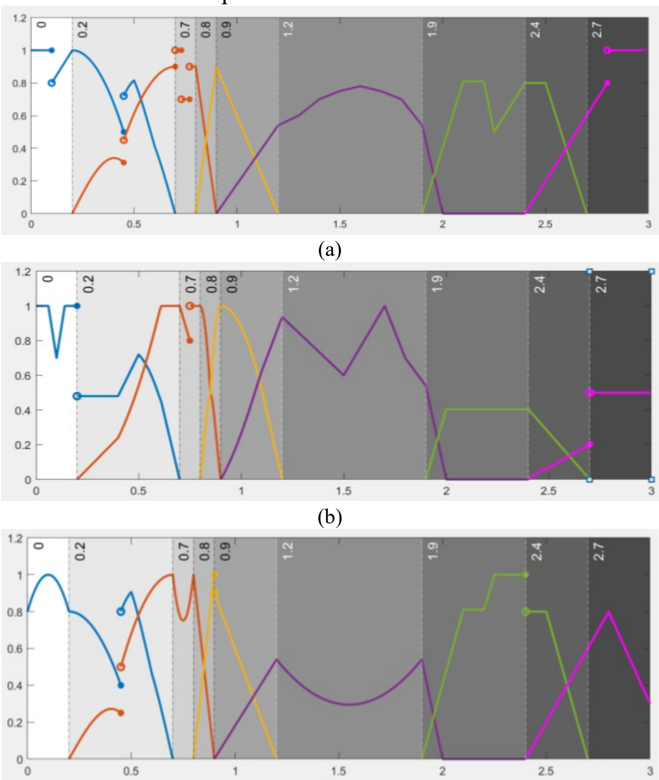
with the same set of fuzzy rules. This implies that convex and normal FMFs are not the only option for developing a monotone TSK-FIS-product model.



(a) (b) (c)

Fig. 14. Three FMF designs of a TSK-FIS-product occurrence model yield the same $\mathcal{P}(x_1)$, for all x_1, with a logarithmic scale [37]

VIII. Conclusions

In this study, we have investigated the properties of TSK-FIS with a "grid partition" strategy to compute the firing strengths with product T-norm, i.e., a TSK-FIS-product model. FMF designs with single and continuous support have been defined. We have formulated a joint necessary condition, whereby each constituent is a necessary condition, for designing a monotone TSK-FIS-product model. Conversely, we have established with counter examples that some conditions, which serve as a joint sufficient condition used to design a monotone TSK-FIS-product model, is not individually required. As such, a joint sufficient condition of a monotone TSK-FIS-product model has been derived. In addition, we have expanded the TSK-FIS-product model, whereby a multi-input TSK-FIS-product model can be treated as a series of single-input TSK-FIS-product-*like* models. Three case studies pertaining to FMEA and image processing have been presented. The results have been compared, analyzed, and discussed, ascertaining the usefulness of our proposed methods.

For further research, the design of FMFs with semantic constraints [51] is suggested. The sufficient condition to construct a monotone TSK-FIS-product model with multiple overlapped FMFs and with unimodal weighting vectors [52] can be conducted. We will also examine monotone type-2 TSK-FIS models [11][53] and multi-output TSK-FIS models in the next phase of research. In addition, designing monotone TSK-FIS models with "*don't care*" conditions, where the firing strengths are computed with min T-norm, will be studied. Investigations on monotone TSK-FIS models utilizing recent advances on fuzzy rule interpolation [16][25][34][35][36][41] are useful. Studies on the relationship between functional OWA [54] and monotone TSK-FIS models will also be carried out.

In addition, inspired by the work in [55], the consideration of the monotone requirement as a design prior, and the practice to relax the monotone requirement with the aim of reducing design conservatism will be formulated as a dilemma for analysis. On the other hand, inspired by the work in [56], the use of the proposed necessary and sufficient conditions for developing monotone generalized fuzzy systems, and/or artificial neural networks (see [22] [57]) will be investigated. Developing monotone TSK-FIS-Product models from data with various learning paradigms [58][59] will also be studied.

Acknowledgement

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APPENDIX

Several relevant properties of FMFs are explained.

Property A. Characteristics of an FMF $\mu_i^{r_i}(x_i)$

A.1. $\mu_i^{r_i}(x_i)$ is *convex* [30] iff for any $d_1, d_2 \in X_i$ and any $\lambda \in [0,1]$, $\mu_i^{r_i}(\lambda d_1 + (1-\lambda)d_2) \geq \min(\mu_i^{r_i}(d_1), \mu_i^{r_i}(d_2))$ is always true. Otherwise, it is *non-convex* [30].

A.2. $\mu_i^{r_i}(x_i)$ is *normal* [53] iff $\exists x_i$ such that $\mu_i^{r_i}(x_i) = 1$. Otherwise, it is *sub-normal* [53].

A.3. $\mu_i^{r_i}(x_i)$ is *jump discontinuous* [53] at $c \in X_i$ if $\lim_{x_i \to c^+} \mu_i^{r_i}(c) \neq \lim_{x_i \to c^-} \mu_i^{r_i}(c)$ where both $\lim_{x \to c^+} \mu_A(c)$ and $\lim_{x \to c^-} \mu_A(c)$ exist.

A.4. $\mu_i^{r_i}(x_i)$ is *complete* if for all x_i, $\mu_i^{r_i}(x_i)$ is defined.