Monotone Fuzzy Rule Interpolation for TSK-FIS-Like n-ary Aggregation Functions

Chian Haur Jong[1], Kai Meng Tay[1*], Yi Wen Kerk[2], and Chee Peng Lim[3]
[1]Faculty of Engineering, Data Science Centre, UNIMAS Water Centre, Universiti Malaysia Sarawak, Kota Samarahan, Malaysia
[2]Center for Artificial Intelligence Technology, Universiti Kebangsaan Malaysia, Bangi, Malaysia
[3]Institute for Intelligent Systems Research and Innovation, Deakin University, Geelong, VIC 3216, Australia
[1]15010109@siswa.unimas.my; [1*]kmtay@unimas.my; [2]ykerk@ukm.edu.my; [3]chee.lim@deakin.edu.au

2023 IEEE International Conference on Fuzzy Systems (FUZZ) | 979-8-3503-3228-5/23/$31.00 ©2023 IEEE | DOI: 10.1109/FUZZ52849.2023.10309818

Abstract*—Fuzzy Rule Interpolation (FRI) is important for fuzzy inference systems modeling pertaining to a sparse fuzzy rule base system. The focus of this paper is on a specific class of FRI, i.e., *monotone FRI* (MFRI), for modeling monotone Takagi-Sugeno-Kang Fuzzy Inference System (TSK-FIS) in the presence of a monotone sparse fuzzy rule base. On the other hand, a function is denoted as an *n*-ary *aggregation function* for a given *n*-dimensional input space and an output space when both the monotone and boundary properties are satisfied. In this paper, a set of *sufficient conditions* derived from the principles of Ordered Weighted Averaging (OWA) and the concept of *orness* for TSK-FIS to obey the monotone property is firstly formulated. We show that it is *necessary* to have a *dense* fuzzy rule base, which can be obtained by interpolation of fuzzy rules in a sparse fuzzy rule base, for constructing a monotone TSK-FIS. We then devise a two-stage MFRI for establishing monotone TSK-FIS. The first stage comprises a *sufficient condition*, inspired from the orness concept, to generate intermediate *fuzzy membership functions* (FMFs). The second stage deduces the monotone consequent of each intermediate rule from the available sparse fuzzy rules. We further extend our MFRI formulation to form TSK-FIS-like *n*-ary aggregation functions.

***Keywords*—monotonicity, fuzzy rule interpolation, Takagi-Sugeno-Kang Fuzzy Inference System, Ordered Weighted Average, orness, Aggregation Functions.**

I. INTRODUCTION

A. Background

Originated from Zadeh [1], reasoning with Fuzzy If-Then rules have been popular for over five decades [2]. A well-known challenge is a *sparse* fuzzy rule base [1] [3], i.e., the fuzzy rule base contains insufficient information pertaining to its total state space. The idea of interpolative reasoning has been proposed to solve issues related to a sparse fuzzy rule base [2], e.g., linear rule interpolation [2] [3] [4]. This further leads to *fuzzy rule interpolation* (FRI)), which is one of the main research topics in the fuzzy community. In general, FRI hinges on the notions of ordering, closeness and distance (see [4]) to *generate intermediate fuzzy rules* for obtaining a dense fuzzy rule base [5]. The important of FRI in fuzzy control has been highlighted, e.g. in [5].

Over the years, various FRI techniques have been proposed. Among them include the interpolation method for triangular membership functions [6], scale and move transformation method [7], similarity transfer interpolation method [8], cutting and transformation-based interpolation method [9]. Other more recent studies on FRI are also reported in [10]-[13]. Nonetheless, research on monotone Fuzzy Rule Interpolation (FRI) is a relatively recent development [14].

B. From Monotone Fuzzy Inference System To TSK-FIS-Like n-ary Aggregation Functions

An FIS, denoted as f, is known as a monotone non-decreasing FIS if it is a mapping $f: X \to Y$ that satisfies $f\left(\boldsymbol{x}_{(1)} = (x_{1,(1)}, \dots, x_{i,(1)}, \dots, x_{n,(1)})\right) \le f\left(\boldsymbol{x}_{(2)} = (x_{1,(2)}, \dots, x_{i,(2)}, \dots, x_{n,(2)})\right)$ for all $x_{i,(1)} \le x_{i,(2)} \in X_i$, $i \in \{1, \dots, n\}$, with an n-dimensional input space $X \in \mathbb{R}^n$ and an output space $Y \in \mathbb{R}$. The consideration of the monotone property as a *prior* requirement has been practiced in FIS modelling [14]-[27]. In general, research studies on monotone-preserving FIS (hereafter denoted as monotone FIS) encompass three aspects: (i) mathematical conditions of an FIS (including interval-type-2 FIS [23]) to satisfy the monotone property for different FIS variants [15]-[27]. including TSK-FIS [17]; (ii) various methods to construct monotone FISs, either via expert knowledge [26]-[27] or data samples; and (iii) various applications of monotone FISs to different domains, including the use of TSK-FIS as *n-ary aggregation functions* [18].

On the other hand, a function, $f_{(agg)}: X \to Y$, is known as an *n*-ary *aggregation function* for a given bounded *n*-dimensional input space, i.e., $X = [0,1]^n$ and an output space, $Y = [0,1]$, when both monotone and boundary properties are satisfied. The monotoney property is defined as $f\left(\boldsymbol{x}_{(1)} = (x_{1,(1)}, \dots, x_{i,(1)}, \dots, x_{n,(1)})\right) \le f\left(\boldsymbol{x}_{(2)} = (x_{1,(2)}, \dots, x_{i,(2)}, \dots, x_{n,(2)})\right)$ for all $x_{i,(1)} \le x_{i,(2)} \in X_i$, $i \in \{1, \dots, n\}$. The boundary property is defined as $f(0,0, \dots 0) = 0$ and $f(1,1, \dots 1) = 1$. Note that a set of sufficient conditions for TSK-FIS to be an *n*-ary aggregation function has been presented [18].

C. Research Gaps and Aims

Accordingly, FRI is critical for TSK-FIS to operate as an *n-ary aggregation function*, since the requirement of a dense fuzzy rule base is necessary for constructing monotone TSK-FIS (see the discussion after *Corollary 1* [18, pp. 1867]). A preliminary scheme for utilizing monotone FRI in practical modelling of the zero-order TSK-FIS has been outlined [14] (see Figure 1). The general idea is to achieve the lowest possible square of the difference between simplified linear FRI-deduced conclusions and MFRI-produced conclusions. A Lagrangian function is adopted, and a convex programming problem is formulated. A unique global optimal solution, which is also the local minimal solution, can be expected using the Karush-Kuhn-Tucker (KKT) optimality conditions.

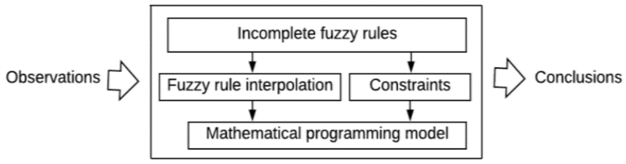


Figure 1 Monotone Fuzzy Rule Interpolation scheme from [14]

The research gaps in the literature and our research aims are outlined as follows.

(1) While monotone TSK-FIS is relevant to ordered weighted averaging (OWA) [28][29] and *orness* [30], it is unclear how the principles of OWA and the concept of orness can be leveraged to construct TSK-FIS. As such, a set of sufficient conditions for TSK-FIS to be monotone is devised based on both OWA and orness. This includes a method to design fuzzy membership functions (FMFs) and monotone consequences.

(2) While MFRI is an important step to generate intermediate fuzzy rules, it is unclear how FMFs of intermediate fuzzy rules should be designed for constructing monotone TSK-FIS with a sparse fuzzy rule base, considering the sufficient condition. In this paper, a method to design FMFs, together with MFRI [14], to generate intermediate fuzzy rules, is presented. With the concept of *hyperbox* [31], we explain the importance of generating intermediate fuzzy rules for establishing *functional* TSK-FIS.

(3) It is unclear how MFRI can be utilised for establishing TSK-FIS with functional consequents and forming TSK-FIS-like *n*-ary aggregation functions. Thus, we illustrate the use of MFRI for TSK-FIS with functional consequence. Also, a set of sufficient condition for TSK-FS to operate line an *n*-ary aggregation function is outlined. The use of MFRI for achieving TSK-FIS like *n*-ary aggregation function is illustrated.

II. Preliminary

A. OWA and Orness

An OWA operator of m dimension is a mapping of F: $\mathbb{R}^m \to \mathbb{R}$ if it has an associated weight vector $\boldsymbol{w} = (w_1, \dots, w_m)^T$ with $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j = 1$, such that $F(a_1, \dots, a_m) = \sum_{j=1}^{m} w_j b_j$, where b_j is the j th largest element a_j. The weights w_j are associated with a particular ordered position b_j rather than with a particular element. An OWA operator is always commutative, monotonic and idempotent [28][29].

Orness of an m-ary OWA is increasing [30], i.e., $Or(\boldsymbol{w}_{(1)}) > Or(\boldsymbol{w}_{(2)})$, if two m-dimensional weight vectors, i.e., $\boldsymbol{w}_{(1)}$ and $\boldsymbol{w}_{(2)}$, satisfy $\boldsymbol{w}_{(1)} = (w_{1,(1)}, \dots, w_{m,(1)})^T$ and $\boldsymbol{w}_{(2)} = (w_{1,(1)}, \dots, w_{\xi} - \epsilon, \dots, w_{\rho} + \epsilon, \dots, w_{m,(1)})^T$, where $\epsilon > 0$ and $\xi < \rho$.

B. TSK-FIS

Consider an TSK-FIS with an input domain $X_i \in X \in \mathbb{R}^n$, $i \in \{1, .., n\}$, the input variable x_i is partitioned into $p_i \geq 1$ FMFs. Each partition is denoted with a linguistic term, $A_i^{r_i}$, with its corresponding FMF $\mu_i^{r_i}(x_i)$, where $r_i \in \{1, \dots, p_i\}$, and $r_1, \dots, r_n$ is an integer. The v^{th} fuzzy If-Then rule of an TSK-FIS has the following form:

R^v: IF x_1 is A_1^v AND …. AND x_n is A_n^v THEN y is $y^v(\boldsymbol{x})$

The firing strength and normalized firing strength of the v^{th} fuzzy If-Then rule, i.e., $R^v: A^v \to y^v(\boldsymbol{x})$ can be obtained with $A^v(\boldsymbol{x}) = \prod_{i=1}^{n} A_i^{r_i}(x_i)$ and $\overline{A^v(\boldsymbol{x})} = \frac{A^v(x)}{\sum_{v=1}^{P} A^v(x)}$, respectively. An TSK-FIS is a mapping of $f: X \to Y$, with P fuzzy If-Then rules, i.e., $f(\boldsymbol{x}) = \frac{\sum_{v=1}^{P} A^v(x) \times y^v(x)}{\sum_{v=1}^{P} A^v(x)} = \sum_{v=1}^{P} \overline{A^v(\boldsymbol{x})}\, y^v(\boldsymbol{x})$.

III. Formation of Hyperboxes of Fuzzy If-Then Rules

Hyperbox is a concept in *fuzzy min-max neural network* models for classification and clustering [31]. An n-dimensional hyperbox is represented by its corresponding minimum and maximum points (i.e., vertices). Here, a hyperbox is formed by the support of antecedents, i.e., *support* of A^v , if it exists, as denoted by $A_0^{v\prime}(\boldsymbol{x})$ ={$(\boldsymbol{x}|A^v(\boldsymbol{x}) > 0)$}.

Support of A^v (denoted as $A_0^{v\prime}$) is defined as an n-dimensional hyperbox confined by its vertices, i.e., the minimum point $\underline{HA^v}$ and maximum point $\overline{HA^v}$. Both $\underline{HA^v}$ and $\overline{HA^v}$ are represented as two n-dimensional vectors:

$\underline{HA^v} = (\underline{x_1^{r_1}}, \underline{x_2^{r_2}}, \dots, \underline{x_n^{r_n}})$

$\overline{HA^v} = (\overline{x_1^{r_1}}, \overline{x_2^{r_2}}, \dots, \overline{x_n^{r_n}})$

such that $\overline{HA^v} \geq \underline{HA^v}$ and $A_0^{v\prime} \in X$. An example of a 2-dimensional hyperbox formed by two FMFs is shown in Figure 2.

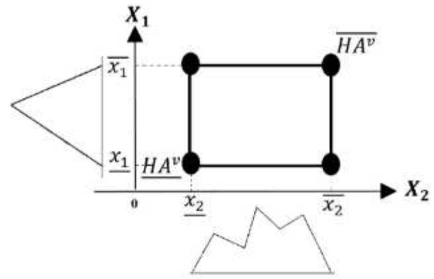


Figure 2. An example of a 2-dimensional hyperbox $A_0^{v\prime} = (\underline{HA^v}, \overline{HA^v})$

We introduce $y^v(\boldsymbol{x}\,|A_0^{v\prime})$ to denote the restriction of $y^v(\boldsymbol{x})$ to $A_0^{v\prime}$. As long as the n-dimensional hyperbox of $y^v(\boldsymbol{x}|A_0^{v\prime})$ is a subset of X, non-monotone $y^v(\boldsymbol{x})$ can produce monotone $y^v(\boldsymbol{x}\,|A_0^{v\prime})$.

IV . OWA and Monotone TSK-FIS

Pertinent to the antecedent and consequent parts of a fuzzy rule base, we outline a set of sufficient conditions for TSK-FIS to be monotone, as follows.

Condition 1

At the antecedent part of a fuzzy rule base, following the orness concept of an OWA operator, all $A_i^{r_i}(x_i)$ are designed such that orness of the vector consisting of r_i numerals, i.e., $\left(\overline{A_i^1(x_i)}, \overline{A_i^2(x_i)}, \dots, \overline{A_i^{p_i}(x_i)}\right)$ increases, when x_i increases.

Condition 2

At the consequent part of a fuzzy rule base, all $y^v(\boldsymbol{x}|A_0^{v\prime})$ are monotone, i.e., $y^v\left(\boldsymbol{x}_{(1)}|A_0^{v\prime} = \left(x_{1,(1)}, \dots, x_{i,(1)}, \dots, x_{n,(1)}\right)\right) \leq y^v\left(\boldsymbol{x}_{(2)}|A_0^{v\prime} = \left(x_{1,(2)}, \dots, x_{i,(2)}, \dots, x_{n,(2)}\right)\right)$ for all $x_{i,(1)} \leq x_{i,(2)} \in$

X_i, and all $v, i \in \{1, \dots, n\}$, but $y^v(\boldsymbol{x})$ may not be monotone entirely.

Note that the proposed set of new sufficient conditions is restricted to the case that the maximal of two FMFs overlap each other at X_i.

V. New Monotone Fuzzy Rule Interpolation for Monotone TSK-FIS With OWA Principles

A. *Sparse fuzzy rule base for monotone TSK-FIS*

The sparse fuzzy rule base problem hinders effective modelling of monotone TSK-FIS when (1) there is an $\boldsymbol{x}$ such that $\sum_{v=1}^{P} A^v(\boldsymbol{x}) = 0$, $\overline{A^v(\boldsymbol{x})}$ is undetermined if 0/0 is undefined; and (2) any $y^v(\boldsymbol{x}|A_0^{v\prime})$ is unknown.

A fuzzy rule base is *dense*, *if and only if* (1) there is no $\boldsymbol{x} \in X$ such that $\sum_{v=1}^{P} A^v(\boldsymbol{x}) = 0$; and (2) all $y^v(\boldsymbol{x}|A_0^{v\prime})$ are defined and known. Note that (1) and (2) are the *necessary and sufficient conditions* for TSK-FIS to satisfy the requirement of having a dense fuzzy rule base.

B. *MFRI for Monotone TSK-FIS*

The use of MFRI for constructing monotone TSK-FIS is presented as a 2-step process, as follows.

B-I. Generating FMF(s) of intermediate fuzzy rules via the orness concept for TSK-FIS

We consider a set of *available fuzzy If-Then rules* for modelling a monotone TSK-FIS. These rules can be obtained from human experts or generated from a data-driven FIS. The available fuzzy If-Then rules satisfy Conditions (1) and (2) specified in Section II, but, *sparse*.

Intermediate fuzzy If-Then rules need to be generated, in such a way that when they are used together with the available fuzzy rules, a monotone TSK-FIS is obtained. Consider the case of p_i FMFs, where each x_i is from the available fuzzy rules, and the associated FMFs are denoted as $A^{r_i}(x_i)$. FMFs for p_i^* intermediate fuzzy rules, denoted as $A^{r_i^*}(x_i)$, should be generated to obtain a dense fuzzy rule base.

p_i^* FMFs for intermediate fuzzy rules are designed in such a way that when x_i increases, orness of $\left(\overline{A^1(x_i)}, \overline{A^2(x_i)}, \dots, \overline{A^{1*}(x_i)}, \overline{A^{2*}(x_i)}, \dots, \overline{A^{p_i^*}(x_i)}, \overline{A^{p_i}(x_i)}\right)$ always increases, see section II-(A). Each pair of $\overline{A^{r_i}(x_i)}$ and $\overline{A^{r_i^*}(x_i)}$ is obtained using:

$$\overline{A^{r_i}(x_i)} = \frac{A^{r_i}(x_i)}{\sum_{r_i=1}^{p_i} A^{r_i}(x_i) + \sum_{r_i^*=1}^{p_i^*} A^{r_i^*}(x_i)} \quad (1)$$

$$\overline{A^{r_i^*}(x_i)} = \frac{A^{r_i}(x_i)}{\sum_{r_i=1}^{p_i} A^{r_i}(x_i) + \sum_{r_i^*=1}^{p_i^*} A^{r_i^*}(x_i)} \quad (2)$$

B-II. Generating consequents for intermediate fuzzy rules via MFRI [14]

With the generated observations, the conclusions can be obtained using the simplified and modified linear interpolative reasoning scheme (linear FRI) [14]. MFRI focuses on the minimization of sum of squared error between the linear interpolative reasoning scheme-deduced conclusion $y^{v*}(\boldsymbol{x})$ and MFRI-produced conclusion $y^{v,rel}(\boldsymbol{x})$, subject to a set of constraints. The purpose is to relabel the non-monotone conclusions deduced by a simplified and modified linear interpolative reasoning scheme, denoted as System (3).

$$\text{Min} \sum_{u=1} (y^{v*}(\boldsymbol{x}) - y^{v*,rel,u}(\boldsymbol{x}))^2 \quad (3.1)$$

subject to

$$y^{v*,rel}(\boldsymbol{x}) \geq \max_{A^{v(t_1 \leq r_1, t_2 \leq r_2, \dots, t_n \leq r_n)*}} y^{v*}(\boldsymbol{x}) \quad (3.2)$$

$$y^{v*,rel}(\boldsymbol{x}) \leq \min_{A^{v(t_1 \geq r_1, t_2 \geq r_2, \dots, t_n \geq r_n)*}} y^{v*}(\boldsymbol{x}) \quad (3.3)$$

$$y^{v(r_1,r_2,\dots,r_n)*,rel_{(1)}}(\boldsymbol{x}) \leq y^{v(r_1,r_2,\dots,r_n)*,rel_{(2)}}(\boldsymbol{x}) \quad (3.4)$$

such that $(r_1, r_2, \dots, r_n)_{(1)} \leq (r_1, r_2, \dots, r_n)_{(2)}$, $\forall\, \boldsymbol{x}$.

C. Example 1: SISO TSK-FIS

An example of an SISO TSK-FIS model ($i = 1$) with a total of $p_1 = 2$ FMFs in the input domain X_1 (as shown in Figure 3) and $p_1 = 2$ fuzzy rules in the $X_1 \rightarrow Y$ domain is considered. The centroid of $A^1(x_1)$ and $A^2(x_1)$ are used for MFRI. Here, $p_1^* = 3$ and $p_1^* = 5$ observations. Both are designed in such a way that vector $\mathcal{H}(x_1)$, i.e., $\mathcal{H}(x_1) = (\overline{A^1(x_1)}, \overline{A^{1*}(x_1)}, \overline{A^{2*}(x_1)}, \overline{A^{3*}(x_1)}, \overline{A^2(x_1)})$ and $\mathcal{H}(x_1) = (\overline{A^1(x_1)}, \overline{A^{1*}(x_1)}, \overline{A^{2*}(x_1)}, \overline{A^{3*}(x_1)}, \overline{A^{4*}(x_1)}, \overline{A^{5*}(x_1)}, \overline{A^2(x_1)})$ increases when x_1 increases.

R^1	IF x_1 is $A^1(x_1)$ THEN y is $y^1 = 1$
R^2	IF x_1 is $A^2(x_1)$ THEN y is $y^2 = 10$

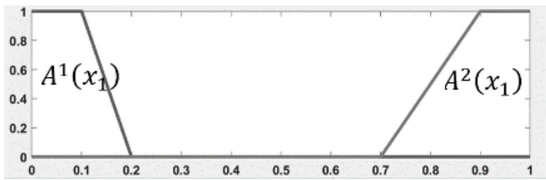
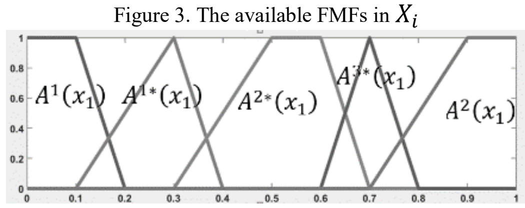
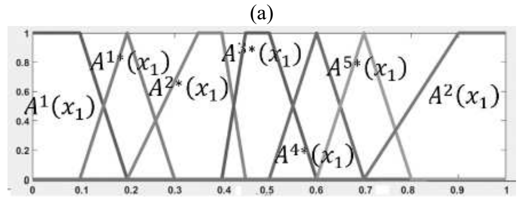


Figure 3. The available FMFs in X_i

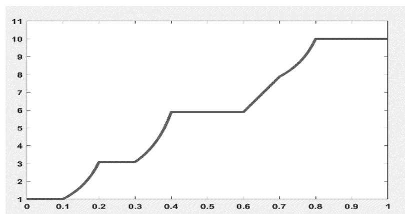


(a)

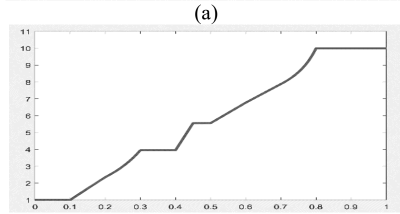


(b)

Figure 4. Designs of FMFs with 3 and 5 observations



(a)



(b)

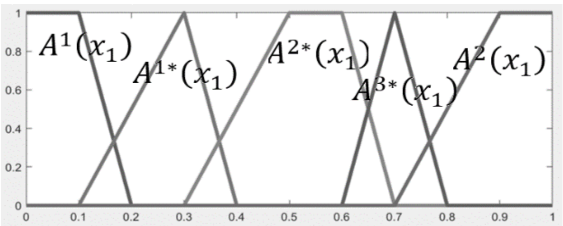
Figure 5 Constructed TSK-FIS with (a) $p_1^* = 3$ and (b) $p_1^* = 5$

Given the designed FMFs shown in Figure 4. (a), with $A^{1*}(x_1)$, $A^{2*}(x_1)$ and $A^{3*}(x_1)$, the deduced outcomes of y^{1*}, y^{2*}, and y^{3*} using linear FRI are 3.0887, 5.8901, and 7.8805, respectively. For the designed FMFs shown in Figure 4 (b), the deduced outcomes of y^{1*}, y^{2*}, y^{3*}, y^{4*}, and y^{5*} using linear FRI are 2.3515, 3.9488, 5.5584, 6.7747, and 7.8805, respectively. Figures 5(a) and 5(b) depict the constructed TSK-FIS with $p_1^* = 3$ and $p_1^* = 5$ observations, respectively.

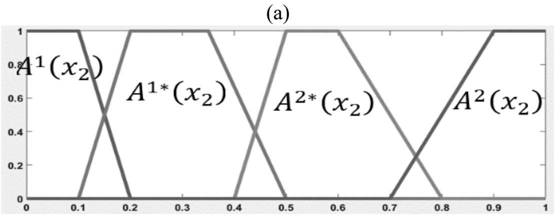
D. Example 2: Two-input TSK-FIS

A two-input TSK-FIS with $p_1 = p_2 = 2$ FMFs in the input domains X_1 and X_2 (as shown in Fig. 3) and $p_1 \times p_2 = 4$ fuzzy rules in the $X_1 \times X_2 \rightarrow Y$ domain is considered. In this example, $p_1^* = 3$ and $p_2^* = 2$ observations, as shown in Figure 6. Both are designed in such a way that vector $\mathcal{H}(\boldsymbol{x})$, i.e., $\mathcal{H}(x_1) = (\overline{A^1(x_1)}, \overline{A^2(x_1)}, \overline{A^{1*}(x_1)}, \overline{A^{2*}(x_1)}, \overline{A^{3*}(x_1)})$ and $\mathcal{H}(x_2) = (\overline{A^1(x_2)}, \overline{A^2(x_2)}, \overline{A^{1*}(x_2)}, \overline{A^{2*}(x_2)})$ increases when x_1 and x_2 increase, where $\boldsymbol{x} = (x_1, x_2)$.

R^1	IF x_1 is $A^1(x_1)$ and x_2 is $A^1(x_2)$ THEN y is $y^1 = 1$
R^2	IF x_1 is $A^1(x_1)$ and x_2 is $A^2(x_2)$ THEN y is $y^2 = 7$
R^3	IF x_1 is $A^2(x_1)$ and x_2 is $A^1(x_2)$ THEN y is $y^3 = 7$
R^4	IF x_1 is $A^2(x_1)$ and x_2 is $A^2(x_2)$ THEN y is $y^4 = 10$



(a)



(b)

Figure 6. Designs of FMFs with 3 and 2 observations

Given the designed FMFs shown in Fig. 6, the deduced $y^{1*,rel}$, ... $y^{16*,rel}$ can be obtained with MFRI (by solving System 3) as in Figure 7. The surface plot for Y versus X_1 and X_2 is depicted in Figure 8.

$A^2(x_1)$	$y^3 = 7$	$y^{14*,rel} = 7.0003$	$y^{15*,rel} = 7.2229$	$y^{16*,rel} = 8.0886$	$y^4 = 10$
$A^{2*}(x_1)$	$y^{9*,rel} = 5.8495$	$y^{10*,rel} = 6.0548$	$y^{11*,rel} = 6.6028$	$y^{12*,rel} = 7.0954$	$y^{13*,rel} = 7.4533$
$A^{1*}(x_1)$	$y^{4*,rel} = 4.0346$	$y^{5*,rel} = 4.7246$	$y^{6*,rel} = 5.8778$	$y^{7*,rel} = 6.4352$	$y^{8*,rel} = 7.0002$
$A^1(x_1)$	$y^1 = 1$	$y^{1*,rel} = 3.8186$	$y^{2*,rel} = 5.5710$	$y^{3*,rel} = 6.3269$	$y^2 = 7$
	$A^1(x_2)$	$A^{1*}(x_2)$	$A^{2*}(x_2)$	$A^{3*}(x_2)$	$A^2(x_2)$

Figure 7. Dense fuzzy rule base with both available fuzzy rules and MFRI generated rules.

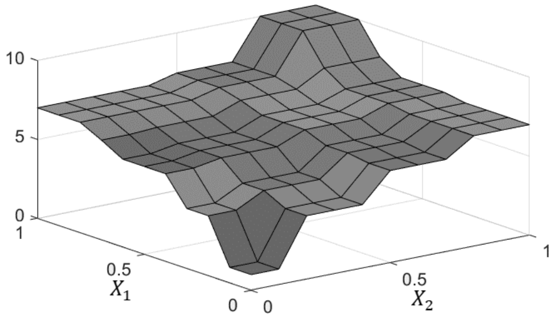


Figure 8: The surface plot for Y versus X_1 and X_2

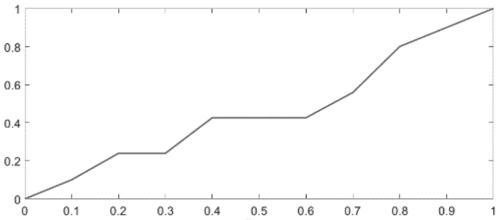
VI. Extension to TSK-FIS with Functional Consequents

Together with the notion of hyberbox (see Section III), the formulation in this study can be extended to TSK-FIS with functional consequents. Consider a modified example presented in Section V-(C), where $\underline{HA^1} = (0)$, $\overline{HA^1} = (0.2)$, and $\underline{HA^2} = (0.7)$, $\overline{HA^2} = (1)$. With the available fuzzy rules in Figure 9, $y^1 = y^2 = x_i$. With hyperbox, $y^1 = [0, 0.2]$, and $y^2 = [0.7, 1]$ are obtained. The same FMFs with intermediate fuzzy rules as in Figures 4 are used for evaluation. The simplest, we consider the average points of $y^1 = [0, 0.2]$ and $y^2 = [0.7, 1]$, for simulation.

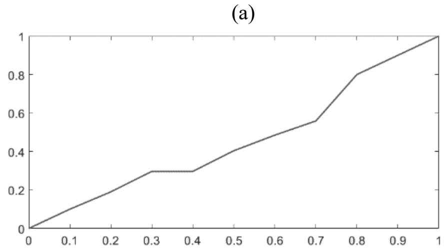
R^1	IF x_1 is $A^1(x_1)$ THEN y is $y^1 = x_i$
R^2	IF x_1 is $A^2(x_1)$ THEN y is $y^2 = x_i$

Figure 9. The available FMFs in X_i with FMFs from Fig. 3.

Given the designed FMFs shown in Figure 4. (a), with $A^{1*}(x_1)$, $A^{2*}(x_1)$ and $A^{3*}(x_1)$, the deduced outcomes of y^{1*}, y^{2*}, and y^{3*} using linear FRI are 0.2392, 0.4260, and 0.5587, respectively. For the designed FMFs shown in Fig. 4 (b), the deduced outcomes of y^{1*}, y^{2*}, y^{3*}, y^{4*}, and y^{5*} using linear FRI are 0.1901, 0.2966, 0.4039, 0.4850, and 0.5587, respectively. Figures 10(a) and 10(b) depict the constructed TSK-FIS with $p_1^* = 3$ and $p_1^* = 5$ observations, respectively.



(a)



(b)

Figure 10. Constructed TSK-FIS with functional consequence with (a) $p_1^* = 3$ and (b) $p_1^* = 5$

VII. MFRI FOR DATA DRIVEN TSK-FIS-LIKE *n*-ARY AGGREGATION FUNCTIONS

A. *Sufficient conditions for TSK-FIS-like n-ary aggregation functions*

For TSK-FIS to operate like an *n*-ary aggregation function, the set of sufficient conditions at both the antecedent and consequent parts of a fuzzy rule base presented in Section IV, is satisfied. In addition, two additional conditions are added, as follows.

(1) $y^v(0, 0, \dots, 0) = 0$, for all v, if $A^v(0,0, \dots 0) > 0$; and

(2) $y^v(1, 1, \dots, 1) = 1$, for all v, if $A^v(1,1, \dots 1) > 0$.

The two additional conditions are introduced in Definitions 3 (see Definition (3.3) and (3.4)) and Proposition 1 in [18] to ensure the boundary properties.

B. *A simulated example*

In this section, a data set consists of 121 input-output data is generated uniformly from a two-input monotonic function, i.e., $y = 0.2e^{x_1} + 0.7e^{x_2}$, and then *normalized* within the interval of [0,1] is considered. A two-input TSK-FIS model is established by considering all the 121 input-output data as the training set. To generate a sparse fuzzy rule base, FMFs as depicted in Fig. 3, is adopted for both x_1 and x_2, respectively. With Wang-Mendel (WM) method [32], a total of four fuzzy rules are obtained is presented in Fig.11. The two conditions from V-A is then imposed to relabel the consequence for R^1 and R^4, in such $y^1 = 0$ and $y^4 = 1$.

R^1	IF x_1 is $A^1(x_1)$ and x_2 is $A^1(x_2)$ THEN y is $y^1 = 0.0306$
R^2	IF x_1 is $A^1(x_1)$ and x_2 is $A^2(x_2)$ THEN y is $y^2 = 0.6931$
R^3	IF x_1 is $A^2(x_1)$ and x_2 is $A^1(x_2)$ THEN y is $y^3 = 0.2199$
R^4	IF x_1 is $A^2(x_1)$ and x_2 is $A^2(x_2)$ THEN y is $y^4 = 0.8824$

Fig 11. Sparse fuzzy rules obtained from WM method

FMFs of intermediate fuzzy rules, for both x_1 and x_2, are designed using the orness concept. Two and three intermediate FMFs are added to x_1 and x_2, respectively. FMFs from Figures 6(a) and 6(b), are adopted, for x_1 and x_2, respectively. A dense fuzzy rule base (See Figure 12) is obtained by solving System (3). A surface plot, with both monotonicity and boundary properties, as depicted in Figure 13, is obtained.

$A^2(x_1)$	$y^3 = 0.2199$	$y^{14*,rel} = 0.6931$	$y^{15*,rel} = 0.6931$	$y^{16*,rel} = 0.6931$	$y^4 = 1$
$A^{2*}(x_1)$	$y^{9*,rel} = 0.2198$	$y^{10*,rel} = 0.4798$	$y^{11*,rel} = 0.4993$	$y^{12*,rel} = 0.5294$	$y^{13*,rel} = 0.6931$
$A^{1*}(x_1)$	$y^{4*,rel} = 0.2199$	$y^{5*,rel} = 0.3286$	$y^{6*,rel} = 0.3857$	$y^{7*,rel} = 0.3880$	$y^{8*,rel} = 0.6931$
$A^1(x_1)$	$y^1 = 0$	$y^{1*,rel} = 0.2198$	$y^{2*,rel} = 0.2199$	$y^{3*,rel} = 0.2199$	$y^2 = 0.6931$
	$A^1(x_2)$	$A^{1*}(x_2)$	$A^{2*}(x_2)$	$A^{3*}(x_2)$	$A^2(x_2)$

Figure 12. A dense fuzzy rule base system obtained by solving System (3)

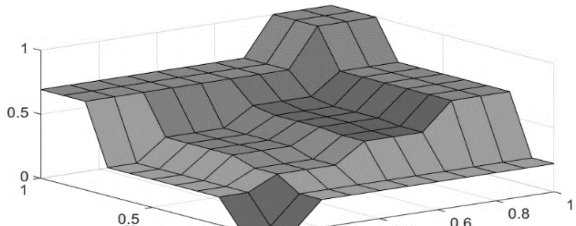


Figure 13. The surface plot of Y versus X_1 and X_2

C. *Comparisons and discussions*

Without FRI, it is impossible to compute all FIS outputs for all $\boldsymbol{x}$, with only a sparse fuzzy rule base from Figure 11 . In other words, some $f(\boldsymbol{x}) \; \forall \; \boldsymbol{x}$ is indeterminate, and the entire surface plot is not obtainable. Therefore, a dense fuzzy rule base is a necessary condition for monotone TSK FIS.

Given intermediate fuzzy rules with a linear FRI scheme, it is possible to deduce all conclusions $y^{v*}(\boldsymbol{x})$. Without a fuzzy rule relabeling process, a monotone fuzzy rule cannot always be obtained (see Figure 14), hence a non-monotone surface is resulted (see Figure 15).

$A^2(x_1)$	$y^3 = 0.2199$	$y^{14*,rel} = 0.3627$	$y^{15*,rel} = 0.5146$	$y^{16*,rel} = 0.6464$	$y^4 = 0.8824$
$A^{2*}(x_1)$	$y^{9*,rel} = 0.3387$	$y^{10*,rel} = 0.3387$	$y^{11*,rel} = 0.4826$	$y^{12*,rel} = 0.5570$	$y^{13*,rel} = 0.6081$
$A^{1*}(x_1)$	$y^{4*,rel} = 0.2575$	$y^{5*,rel} = 0.3244$	$y^{6*,rel} = 0.4522$	$y^{7*,rel} = 0.5314$	$y^{8*,rel} = 0.5832$
$A^1(x_1)$	$y^1 = 0.0306$	$y^{1*,rel} = 0.2641$	$y^{2*,rel} = 0.4417$	$y^{3*,rel} = 0.5486$	$y^2 = 0.6931$
	$A^1(x_2)$	$A^{1*}(x_2)$	$A^{2*}(x_2)$	$A^{3*}(x_2)$	$A^2(x_2)$

Figure 14. A dense fuzzy rule base system acquired through a linear FRI approach.

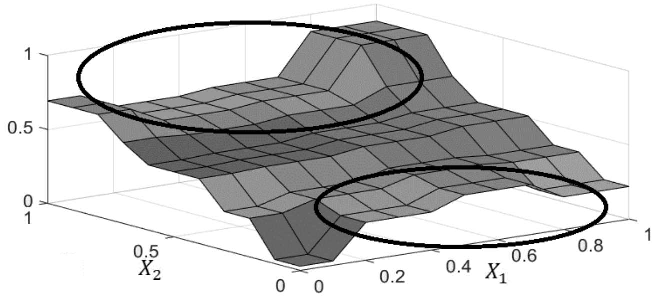


Figure 15. The surface plot of Y versus X_1 and X_2 with a dense fuzzy rule base obtained using a linear FRI method.

Consider System (3) without the sufficient conditions in VII(A), a monotone fuzzy rule can be obtained (see Figure 16), resulting in a monotone surface plot (see Figure 17). However, the boundary properties are not satisfied.

$A^2(x_1)$	$y^3 = 0.2199$	$y^{14*,rel} = 0.6931$	$y^{15*,rel} = 0.6931$	$y^{16*,rel} = 0.6931$	$y^4 = 0.8824$
$A^{2*}(x_1)$	$y^{9*,rel} = 0.2199$	$y^{10*,rel} = 0.4798$	$y^{11*,rel} = 0.4990$	$y^{12*,rel} = 0.5298$	$y^{13*,rel} = 0.6930$
$A^{1*}(x_1)$	$y^{4*,rel} = 0.2199$	$y^{5*,rel} = 0.3282$	$y^{6*,rel} = 0.3857$	$y^{7*,rel} = 0.3881$	$y^{8*,rel} = 0.6930$
$A^1(x_1)$	$y^1 = 0.0306$	$y^{1*,rel} = 0.2199$	$y^{2*,rel} = 0.2199$	$y^{3*,rel} = 0.2199$	$y^2 = 0.6931$
	$A^1(x_2)$	$A^{1*}(x_2)$	$A^{2*}(x_2)$	$A^{3*}(x_2)$	$A^2(x_2)$

Figure 16. A dense fuzzy rule base system acquired with System (3) without sufficient conditions in VII(A)

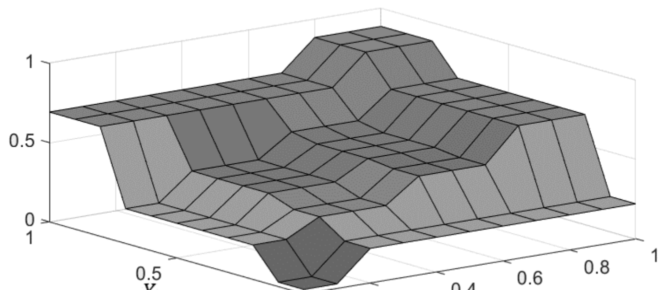


Fig 17. The surface plot of Y versus X_1 and X_2

D. *Remarks*

Instead of imposing hard rules to relabeling the consequent to 0 or 1, by following the sufficient conditions in Section VII (A) and then solving System (3). Another alternative maybe System (4), which can ensure both monotone and boundary properties, as follows.

Min $\sum_{u=1}(y^{v*}(\boldsymbol{x}) - y^{v*,rel,u}(\boldsymbol{x}))^2$ (4.0)

subject to (3.2), (3.3), (3.4) and where

$y^v(0,0,\ldots,0) = 0, \forall\, v$, if $A^v(0,0,\ldots 0) > 0$ (4.1)

$y^v(1,1,\ldots,1) = 1, \forall\, v$, if $A^v(1,1,\ldots 1) > 0$ (4.2)

such that $(r_1, r_2, \ldots, r_n)_{(1)} \leq (r_1, r_2, \ldots, r_n)_{(2)}, \forall\, \boldsymbol{x}$.

VIII. CONCLUSIONS

In this paper, a new sufficient condition for TSK-FIS to be monotone is outlined, considering OWA and orness. This includes a method to design FMFs, considering the orness concepts. It is important to a fuzzy rule base to be dense, for obtaining a monotone TSK-FIS. As a solution to sparse fuzzy rule base system, a MFRI is outlined. This includes a method to generate intermediate fuzzy rules, i.e., to generate FMFs with orness concept, and a simplified interpolator with optimization. We further borrow the concept of hyperbox, allowing the functional consequence, for TSK-FIS, to be represented as an interval, then, the MFRI could be used to generate numerals. In addition, we also outline a set of condition, for TSK-FIS to operate as an *n*-ary aggregations function. Again, the use of MFRI to realize data-driven *n*-ary aggregations function, is illustrated.

For further research, we will conduct detailed studies on new formulation of System (4) and its application to real-world problems.

ACKNOWLEDGEMENT

This work was supported by the Fundamental Research Grant Scheme under Grant FRGS/1/2020/ICT02/UNIMAS/02/2, by the Ministry of Higher Education, Malaysia.

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