A COMPLEX VARIABLE BOUNDARY ELEMENT METHOD FOR TWO-DIMENSIONAL THERMAL ANALYSIS IN ANISOTROPIC SOLIDS

ABDUL RAHMAN B. HJ. ABDILLAH

Universiti Malaysia Sarawak
1999

A Complex Variable Boundary Element Method For Two-Dimensional Thermal Analysis in Anisotropic Solids

By

Abdul Rahman B Hj Abdillah

A report submitted in partial fulfilment of the requirements for the degree of Bachelor of Engineering (Hons.)

Faculty of Engineering

UNIVERSITY MALAYSIA SARAWAK

May 1999

Borang Penyerahan Tesis
Universiti Malaysia Sarawak

R13a

BORANG PENYERAHAN TESIS

Judul : A COMPLEX VARIABLE BOUNDARY ELEMENT METHOD (CVBEM) FOR TWO - DIMENSIONAL THERMAL ANALYSIS IN ANISOTROPIC SOLIDS

SESI PENGAJIAN : 96 - 99

Saya ABDUL RAHMAN B. HJ ABDILLAH

Mengaku membenarkan tesis ini disimpan di Pusat Khidmat Maklumat Akademik, Universiti Malaysia Sarawak dengan syarat-syarat kegunaan seperti berikut :

Hakmilik kertas projek adalah di bawah nama penulis melainkan sebagai projek bersama dan dibiayai oleh UNIMAS, hakmiliknya adalah kepunyaan UNIMAS.

Naskhah salinan di dalam bentuk kertas atau mikro hanya boleh dibuat dengan kebenaran bertulis daripada penulis.

Pusat Khidmat Maklumat Akademik, UNIMAS dibenarkan membuat salinan pengajian mereka.

Kertas projek hanya boleh diterbitkan dengan kebenaran penulis. Bayaran royalti adalah mengikut kadar yang dipersetujui kelak.

* Saya membenarkan / ~~tidak membenarkan~~ Perpustakaan membuat salinan kertas projek ini sebagai bahan pertukaran di antara institusi pengajian tinggi.

** Sila tandakan (✓)

☐	SULIT	(Mengandungi maklumat yang berdarjah keselamatan atau kepentingan Malaysia sperti yang termaktub di dalam AKTA RAHSIA RASMI 1972)
☐	TERHAD	(Mengandungi maklumat TERHAD yang telah ditentukan oleh organisasi/badan di mana penyelidikan dijalankan).
✓	TIDAK TERHAD	

Disahkan oleh

(TANDATANGAN PENULIS)
Alamat tetap : 161, Jalan Kampung Haji Baki, 93250 Kuching Sarawak

Tarikh 5 - 6 - 1999

(TANDATANGAN PENYELIA)

Associate Professor W. T. Ang
(Nama Penyelia)

Tarikh 17/6/99

CATATAN
* Potong yang tidak berkenaan
** Jika Kertas Projek ini SULIT atau TERHAD, sila lampirkan surat daripada pihak berkuasa/organisasi berkenaan dengan menyertakan sekali tempoh kertas projek. Ini perlu dikelaskan sebagai SULIT atau TERHAD

APPROVAL SHEET

This project report attached hereto, entitled "A Complex Variable Boundary Element Method for Two-Dimensional Thermal Analysis in Anisotropic Solids" prepared and submitted by Abdul Rahman B Hj Abdillah in partial fulfillment of the requirement of Bachelor's degree with Honours in Mechanical Engineering is hereby accepted.

______________________________ Date 17/6/99

(Assoc. Prof. Dr. W. T. Ang)

Project Supervisor

DECLARATION

No portion of the work referred to in this report has been submitted in support of an application for another degree or qualification of this or any other university or institution of higher learning.

ABDUL RAHMAN HJ ABDILLAH

5/6/99

(Date)

ACKNOWLEDGEMENTS

ALHAMDULILLAH the highest gratitude can only be and only can be given to THE ONE, Allah s.w.t. Really I don't know what other way to thank You... apart from being Your true humble slave.

Many thanks to the next best things in life, my mom, Hajjah Saadiah, my brothers and sisters, especially Salleh, Faisal, Alauyah and Siti. Without the presence of these significant roles in my life, I would not have the opportunity to thank the next most important person for the completion of this project, that is my supervisor, Associate Professor Dr.W.T. Ang. Dr. Ang, I sincerely thank you for generously contributing ideas, persistent in giving support as well as guidance and patience in putting up with first time researchers like myself. There is no way that I can ever repay the things that I have now...To all my friends, especially Alam, Satria, Pendi, Yayah, Salina, Iwan, Hendison, Dollah, Gaban, Mail, Amco, Taip, Andy and many more... thank you for everything... for being there when I need somebody... for showing me ways when I am lost... May Allah bless all of you. Again, I'm truly blessed to have all of you around me as companions......Endless Thank You to all........

Wassalam

TABLE OF CONTENTS

ABSTRAK

Di dalam projek ini, kaedah pembolehubah nombor kompleks digunakan untuk menyelesaikan masalah konduksi haba yang melibatkan titik sempadan. Kaedah yang digunakan adalah berdasarkan kaedah kamiran Cauchy. Satu aturcara telah dibangunkan dengan menggunakan bahasa pengaturcaraan FORTRAN 77. Aturcara ini digunakan untuk mengira taburan haba dan medan haba dalam sesuatu pepejal.

ABSTRACT

In the present project, the complex variable boundary element method (CVBEM) is applied to solve some heat conduction problems in solid. The method is based on the Cauchy integral formulae. A computer program in FORTRAN 77 based on the CVBEM is developed to calculate temperature and heat flux distribution in solid.

CHAPTER 1

MATHEMATICAL PRELIMINARIES

1.1 Introduction

Heat conduction is increasingly important in modern technology, in the earth sciences, and in many other evolving areas of thermal analysis. The specification of temperatures, of heat sources, and heat flux in regions of material in which conduction occurs, gives rise to analyses of temperature distributions, heat flows, and conditions of thermal stressing. The importance of such conditions has led to an increasingly highly developed analysis in which sophisticated mathematical and increasingly powerful numerical techniques are used.

The main objective of the project is to apply the complex variable boundary element method (CVBEM) to calculate the temperature distribution in anisotropic body. The CVBEM is implemented in FORTRAN code.

In order to devise a numerical method for solving solve the heat conduction problem, it is essential to understand the physics and mathematics, which is relevant to the heat conduction in a solid. In the present chapter, the basic equations for heat conduction in anisotropic bodies are presented. The boundary value problem (BVP) for two-dimensional analysis of steady-state temperature in anisotropic bodies is briefly explained. A general discussion of the boundary element method (BEM) for solving the BVP is also given.

1.2 Fourier's Law

The early development of heat conduction is largely due to the effort of French mathematician, Joseph Fourier. In 1822, he proposed what is well known today as Fourier's law of heat conduction, i.e,

$$q_i = -\lambda(\partial T/\partial x_i) \tag{1.1}$$

In (1.1) q_i is the heat flux vector (w/m^2) in x_i direction and x_i could be x,y or z in a three dimensional Cartesians. T is the temperature (K) and λ is the thermal conductivity of the material (w/mK). Note that the minus sign in (1.1) implies that heat flows from a point with higher temperature to a point with lower temperature [2].

The equation for the heat flux as given by (1.1) is valid for thermally isotropic bodies. Roughly speaking, the term "thermally isotropic" mean that if a heat source is placed at a point in a solid there is no preferred direction of heat flow.

In the case of anisotropic materials, there is a preferred direction of heat flow. Therefore, Fourier's law as given by (1.1) needs to be modified in order to include the directional dependence of the thermal conductivity. The modification of (1.1) for Fourier's law in anisotropic material is as follows:

$$q_i = -\sum_{j=1}^{3} \lambda_{ij} \frac{\partial T}{\partial x_j} \quad , i = 1,2,3 \tag{1.2}$$

Where λ_{ij} are the coefficients describing the thermal conductivity of the solid. From the laws of physics, it is well known that the coefficients λ_{ij} have to satisfy the properties

$$\lambda_{ij} = \lambda_{ji} \tag{1.3}$$

and

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_{ij} \xi_i \xi_j > 0 \text{ for any non - zero matrix } [\xi_i] \tag{1.4}$$

Note that for thermally isotropic bodies, $\lambda_{ij} = \lambda \delta_{ij}$, where δ_{ij} is the Kronecker delta.

1.3 Governing Equation for Conservation of Energy

The flow of heat in a solid must obey the first law of thermodynamic, which is also known as the law of conservation of energy. The first law of thermodynamic states that the rate of energy flowing into a closed system is equal to the rate of energy flowing out, plus the rate at which energy accumulates inside the system [6]. For further details on the above law, refer to Holman [6]. Mathematically, the law can be expressed as (in the absence of any internal heat generator)

$$\sum_{i=1}^{3} \frac{\partial q_i}{\partial x_i} = \rho c \frac{\partial T}{\partial t} \tag{1.5}$$

where ρ and c are density and specific heat capacity of the solid respectively.

In this project the temperature is assumed to be dependent on only x_1 and x_2, i.e. two-dimensional and steady state. The body is assumed to be homogeneous so that the coefficient λ_{ij} are constant. With these assumptions, substituting (1.2) into (1.5) gives

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\lambda_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j}=0 \tag{1.6}$$

Notice that for the two-dimensional analysis, (1.4) can be written as

$$\lambda_{11}\lambda_{22}-\lambda_{12}^2>0 \tag{1.7}$$

The two-dimensional analysis is valid for structures in the form of a very thin flat plate or for an infinitely long cylinder whose geometry does not change in the x_3 axis. The geometry is shown in Figure 1.1. In the former, the plate is thermally insulated everywhere except possibly along its edge. In the latter, temperature or heat flux independent of x_3 is prescribed on the boundary of the infinitely long cylinder.

1.4 Boundary Value Problem

The steady-state two-dimensional thermal analysis can be mathematically formulated in terms of a BVP. Let the body be denoted by R and its boundary by C as in Figure 1.2. The BVP is to solve (1.6) in R subject to

$$\left.\begin{aligned} T(x_1,x_2)&=\Phi(x_1,x_2) \quad \text{for } (x_1,x_2)\in C_1 \\ P(x_1,x_2)&=Q(x_1,x_2) \quad \text{for } (x_1,x_2)\in C_2 \end{aligned}\right\} \tag{1.8}$$

where $\varphi(x_1, x_2)$ and $Q(x_1, x_2)$ are suitably prescribe function of x_1 and x_2 and P is the heat flux across the curve C defined by $P=\lambda_{kj}n_k\partial T/\partial x_j$, with n_k being components of the unit outer normal vector to R on C. C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$.

1.5 Boundary Element Method (BEM)

The BVP above can be exactly solved for simple cases only, e.g. square boundary and simple forms of ϕ and Q. In general, it has to be solved numerically or approximately. One numerical method for solving the BVP is the so-called boundary element method (BEM). In the BEM, a boundary integral solution of (1.6) is obtained. The trick is then to use the integral solution to satisfy the condition (1.8). The method works for any general shape of C and for any suitably prescribed function ϕ and Q.

The main advantages of the BEM, compared to other numerical techniques such as finite element method and finite difference method, can be summarized in the following statements:

- Modeling boundaries and boundary conditions very natural.
- The BEM is ideal for problems with infinite domains, such as problems of soil mechanics, fluid mechanics and thermal analysis and elastodynamic.
- The BEM offers a fully continuous solution inside the domain, and the problem parameters can be evaluated directly at any point there.
- Resolution of response gradients is not tied to volume mesh refinement, as the case would be in the finite element method.

The BEM of course has its own drawbacks, which have some effect in making it less popular with engineers than other methods, and they can be outlined as follows:

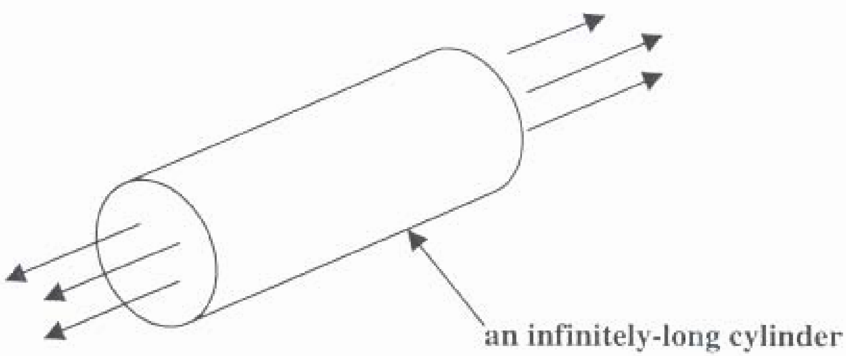
- The derivation of the governing boundary integral equations (BIEs) may require a level of mathematics higher than that with other methods, but the procedure of the BEM itself is no different from that of the finite element Method, FEM.
- The BIEs of nonlinear problems may have domain integrals, which require the use of domain elements for their evaluation, thus losing the main advantage of the dimensionality reduction mentioned above. It must be mentioned though that in recent years advances in research have been able to deal quite successfully with the issue of domain integrals that pop up in nonlinear problems.

For further details on the above advantages and disadvantages of BEM, refer to James H.Kane [5].

More traditional approach of the BEM makes use of real analysis- with Green's theorem and fundamental solution. For further detail, see El-Zafrany [1]. For two-dimensional problems, there is an alternative to the traditional BEM approach, i.e. to use complex analysis with the Cauchy integral formulae. BEM based on complex analysis is known as the complex variable boundary element method (CVBEM).

The CVBEM was originally introduced by Hromadka II and Lai [4] for solving boundary value problems governed by two-dimensional Laplace's equation. In the present project a version of the CVBEM which was recently proposed by Ang and Park [9] will be used to solve the two-dimensional heat conduction problem. The CVBEM approach of Ang and Park [9] differs from that of Hromadka and Lai [4] in the manner in which the boundary conditions are treated. The CVBEM of Ang and Park [9] is derived in Chapter 2 and tested using a BVP with a known solution. In chapter 3, the CVBEM is applied to solve specific heat conduction problems.

(a)



(b)

Figure 1.1: (a) a thin plate; (b) an infinitely-long cylinder

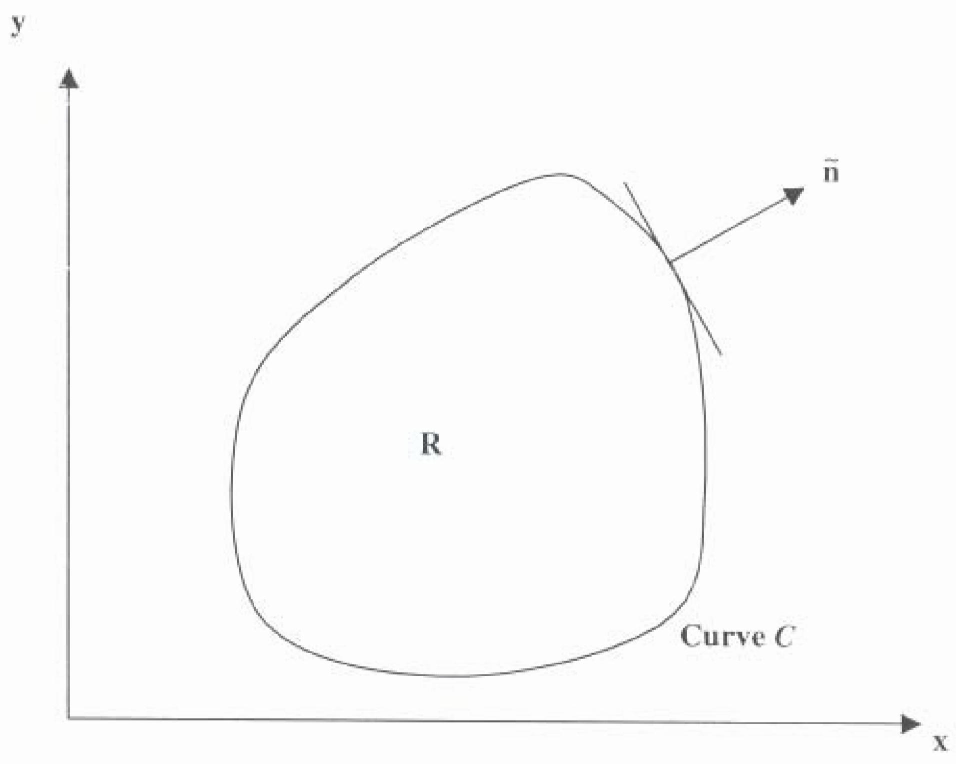


Figure 1.2: Boundary with simple close curve *C*

CHAPTER 2

COMPLEX VARIABLE BOUNDARY ELEMENT METHOD

2.1 Introduction

In this chapter, a boundary element method based on the Cauchy's integral formulae, called the complex boundary element method (CVBEM), which was introduced by Ang and Park (9), for solving the BVP in section (1.8) will be explained in details. The method reduces the BVP to a system of linear algebraic equations. The coefficients of the algebraic equation are easy to compute. Hence it is easy to implement the method on a computer. The derivation of the method is done by Ang and Park [9] and Ang, Clements and Cooke [10]. It is repeated in this chapter.

2.2 CVBEM

It is well known that the real and imaginary parts of a complex function $f(z)=f(x_1 + \tau x_2)$ which is analytic in the region R are the solution of the system (1.6) in R. Therefore, the system (1.6) has the general solution of the form

$$T(x_1, x_2) = \mathrm{Re}\{f(x_1 + \tau x_2)\} \tag{2.1}$$

where $\tau = \dfrac{-\lambda_{12} + i\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^{\,2}}}{\lambda_{22}}$, $i = \sqrt{-1}$

Since f is analytic function of $x_i + \tau x_2$ for $(x_i + x_2) \in$ RUC the Cauchy's integral formula to be used are given by

$$f(c) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - c} \tag{2.2}$$

$$f'(c) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - c)^2} \tag{2.3}$$

Where $c = \xi_1 + \tau\xi_2$ and ξ_1 and ξ_2 are real parameters and C is assigned an anticlockwise direction and $z = x_1 + \tau x_2$.

The boundary C is discretised by placing M points $(x_1^{(1)}, x_2^{(1)}), (x_1^{(2)}, x_2^{(2)}), \ldots\ldots\ldots\ldots x_1^{(m)}, \tau x_2^{(m)})$ (in anticlockwise direction) on it. If we denote the straight line segment from $(x_1^{(k)}, x_2^{(k)})$ to $(x_1^{(k+1)}, x_2^{(k+1)})$ by $C^{(k)}$ [k=1,2,.......M] and $(x_1^{(m+1)}, x_2^{(m+1)}) = (x_1^{(1)} + \tau x_2^{(1)})$. Now we can make an approximation

$$C = C^{(1)} \cup C^{(2)} \cup C^{(3)} \cup \ldots\ldots\ldots \cup C^{(m)} \tag{2.4}$$

with the approximation (2.4), the formula (2.2) may be approximately written as:

$$f(c) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{c^{(k)}} \frac{f(z)dz}{z - c} \tag{2.5}$$

To evaluate the complex integral over $C^{(k)}$ in (2.5), let us expand $f(z) = f(x_1 + \tau x_2)$ as a Taylor Maclaurin series about $z_o^{(k)}$ where $z_o^{(k)} = y_1 + \tau y_2$ and $(y_1^{(k)}, y_2^{(k)})$ is the midpoint of $c^{(k)}$. Therefore,

$$f(z) = f(z_o^{(k)}) + (z - z_o^{(k)}) f'(z_o^{(k)}) + \tfrac{1}{2}(z - z_0^{(k)})^2 f''(z_0^{(k)}) + \ldots\ldots \tag{2.6}$$

$$\theta(\xi_1,\xi_2)=\cos^{-1}\left\{\frac{\left|z^{(k+1)}-c\right|^2+\left|z^{(k)}-c\right|^2-\left|z^{(k+1)}-z^{(k)}\right|}{2\left|z^{(k+1)}-c\right|\left|z^{(k)}-c\right|}\right\} \tag{2.11}$$

If the point c is pushed to approach $z_0^{(k)}$ from within R, we find that (2.9) still remains valid and can be written as

$$\sum_{k=1}^{m}(u^{(k)}+iv^{(k)})\left[\gamma^{(k)}(y_1^{(m)},y_2^{(m)})+i\theta^{(k)}(y_1^{(m)},y_2^{(m)})-2\pi i\delta_{km}\right]=0 \tag{2.12}$$

Where δ_{km} is the kronecker-delta and $(y_1^{(m)},y_2^{(m)})$ is the midpoint of $C^{(m)}$. The real and imaginary part of (2.12) will become

$$\sum_{k=1}^{m}\left\{v^{(k)}\left[2\pi i\delta_{km}-\theta^{(k)}(y_1^{(m)},y_2^{(m)})\right]+u^{(k)}\gamma^{(k)}(y_1^{(m)},y_2^{(m)})\right\}=0, \tag{2.13}$$

$$\sum_{k=1}^{m}\left\{u^{(k)}\left[\theta^{(k)}(y_1^{(m)},y_2^{(m)})-2\pi i\delta_{km}\right]+v^{(k)}\gamma^{(k)}(y_1^{(m)},y_2^{(m)})\right\}=0. \tag{2.14}$$

It seems that either (2.13) and (2.14) can be used to set up a system of linear algebraic equation with k = 1,2,….,M. Notice that $\gamma^{(k)}(y_1^{(k)},y_2^{(k)})=0$ and $\theta^{(k)}(y_1^{(k)},y_2^{(k)})=\pi$. Since $\gamma^{(k)}(y_1^{(k)},y_2^{(k)})=0$, choosing (2.14) may result to an ill-conditioned system. Thus we prefer (2.13).

The system (2.13) y consists of M equation but there are 2M unknown $u^{(k)}$ and $v^{(k)}$. In order to solve the algebraic equation, more equation is needed. The other equation come from the boundary condition.

As mentioned before, on each elements of $C^{(k)}$ either Φ or $n.\nabla\Phi$ is specified. From (1.8)

$$u^{(k)} = \Phi(y_1^{(k)}, y_2^{(k)}) \quad \text{if T is specified over } C^{(k)} \tag{2.15}$$

The above equation can be rewritten as

$$\sum_{m=1}^{M} \delta_{km} u^{(m)} = \phi\left(y_1^{(m)}, y_2^{(m)}\right) \tag{2.16}$$

Now let us consider the boundary condition where P is specified over $C^{(k)}$. As mentioned before, P is given as:

$$P = \lambda_{kj} n_k \frac{\partial T}{\partial x_j} \tag{2.17}$$

T is a real function. From (2.1) where $T = \text{Re}\{f(x_1 + \tau x_2)\}$

It follow that

$$\frac{\partial T}{\partial x_1} = \text{Re}\{f'(x_1 + \tau x_2)\}$$

and $$\frac{\partial T}{\partial x_2} = \text{Re}\{\tau f'(x_1 + \tau x_2)\} \tag{2.18}$$

Now, let we expand (2.17)

$$\sum_{i=1}^{2} n_i \left[\lambda_{i1} \frac{\partial T}{\partial x_1} + \lambda_{i2} \frac{\partial T}{\partial x_2}\right] = Q(x_1, x_2) \tag{2.19}$$

By putting (2.18) into (2.19), it will give us

$$\sum_{i=1}^{2} n_i \left[\lambda_{i1} \operatorname{Re}\{f'(x_1 + \tau x_2)\} + \lambda_{i2} \operatorname{Re}\{\tau f'(x_1 + \tau x_2)\}\right] = Q(x_1, x_2)$$

$$\sum_{i=1}^{2} n_i \operatorname{Re}\{[(\lambda_{i1} + \tau\lambda_{i2}) f'(x_1 + \tau x_2)]\} = Q(x_1, x_2) \qquad (2.20)$$

Assume that $L_i = \lambda_{i1} + \tau\lambda_{i2}$, so (2.20) can be rewritten as

$$\operatorname{Re}\left\{\sum_{i=1}^{2} n_i L_i f'(x_1 + \tau x_2)\right\} = Q(x_1, x_2) \qquad (2.21)$$

To deal with condition (2.21), we use the Cauchy integral formula. From (2.5), if we use Taylor-Maclaurin series

$$\int_{C^{(k)}} \frac{f(z)dz}{(z-c)^2} = f(z_o^{(k)}) \int_{C^{(k)}} \frac{dz}{(z-c)^2} + f'(z_o^{(k)}) \int_{C^{(k)}} \frac{(z - z_o^{(k)})dz}{(z-c)^2} +$$
$$\tfrac{1}{2} f''(z_o^{(k)}) \int_{C^{(k)}} \frac{(z - z_o^{(k)})dz}{(z-c)^2} + \ldots\ldots \qquad (2.22)$$

By omitting $0(h^{(k)}]^2)$ or higher order terms in (2.22) w find that

$$\int_{C^{(k)}} \frac{f(z)dz}{(z-c)^2} = f(z_o^{(k)}) \left[\frac{-1}{z^{(k+1)} - c} + \frac{1}{z^{(k)} - c}\right] \quad \text{if } c \neq z_o^{(k)} \qquad (2.23)$$

If we let c approaches $z_0^{(k)}$ from within R in (2.21) and neglect O[h(k)] or higher order terms, we obtain

$$\lim_{c \to z_o^{(k)}} \int_{C^{(k)}} \frac{f(z)dz}{(z-c)^2} \approx f(z_o^{(k)}) \left[\frac{-1}{z^{(k+1)} - z_0^{(k)}} + \frac{1}{z^{(k)} - z_0^{(k)}}\right] + \pi i f'(z_o^{(k)}) \qquad (2.24)$$